Comparative Dynamics of the Standard Real Business Cycle Model

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Despite numerous efforts at interpretation and exposition, the standard Real Business Cycle model remains somewhat mysterious. Its dynamic properties are by now familiar, at least to those who made the effort to run simulations by themselves. Many extensions of the basic model have been developed, which considerably enrich its mechanisms. Yet understanding all these mechanisms usually draws more on intuition rather than on formal and detailed analysis. This may be largely unavoidable in an approach based on simulation, which values in the first place the ability to reproduce stochastic properties of actual time series. Our opinion however is that it is useful to develop simultaneously an analytic evaluation of the various models. Using all the tools of economic theory should enable us to reach a more precise understanding of the mechanisms at work, and therefore to better assess the robustness of simulation results.

Many authors have already adopted this perspective, demonstrating in effect that the boundary between analytical and simulation studies may become rather fuzzy. King(1991) forcefully advocated a systematic use of the microeconomic distinction between substitution and wealth effects to elucidate the dynamic properties of real business cycle models. Fairise(1994) pursued this approach and provided a detailed analysis in a paper which bears some resemblance with our own. Following a different line, Campbell(1994) also offers a detailed analytical study of the standard real business cycle model. Our work shares the same spirit.

We follow nevertheless a somewhat indirect route since we shall describe the comparative dynamics of the non-stochastic growth model with endogenous labor supply. We study the behavior of the economy in the neighborhood of the balanced growth path, when it is struck by deterministic technological shocks. It is well-known, since the work of King-Plosser-Rebelo(1988), that log-linearization of a real business cycle model around a deterministic growth path leads to a satisfactory approximation of the true solution. It is therefore not surprising that our analysis yields basically the same results as the standard approach using a stochastic model, and we shall have several opportunities to demonstrate this similarity.

Another peculiarity of our work is that we use a continuous time model. Although a discrete time model would yield very similar results, a continuous time model has some advantages. Calculations are more elegant. More importantly, continuous time is the only way to model rigorously a purely instantaneous shock. As we shall see, this will enable us to clarify the respective roles of substitution and income effects. To perform our comparative dynamics analysis, we shall use in a systematic way a simple variational calculus, the elements of which are the relative deviations from the balanced growth path. There is no real contribution to methodology here, as the mathematical method is well-known and has been developed for instance by Oniki(1973) and Aoki(1980). Our aim however is to show how natural this calculus is for an economist, and how powerful it is when one wishes to interpret in detail the comparative dynamics of the model.

Our analysis parallels the usual treatment of a real business cycle model. We loglinearize the model around the balanced growth path and we solve it to obtain explicit impulse response functions. We offer a general analysis of initial responses of consumption and labor supply and we provide, in the case of permanent shocks, an analytical regioning of the parameter space. We examine in detail the influence of the degree of persistence of technological shocks. We also use impulse functions to compute indicators of the co-movements of various variables. Using the calibration of King-Plosser-Rebelo(1988), we then obtain numerical results which are surprisingly close to their own results. This demonstrate that the use of a deterministic, continuous-time model has very little effect on the characteristic properties of the model.

We then develop an economic interpretation of these results in terms of income and substitution effects. This distinction applies in principle to the behavior of a pure consumer facing given wages and interest rates. We therefore turn to the decentralized version of the model to obtain a precise decomposition of the various effects of a technological shock. We are thus able to distinguish clearly and to quantify the effects which have been alluded to repeatedly in the literature, starting from the pionneering analysis of King-Plosser-Rebelo(1988).

The model

We consider a standard growth model, with endogenous labor supply and Harrod-neutral autonomous technical progress.

$$\begin{cases} Max \int_0^{+\infty} e^{-\rho t} U(c_t, l_t)\ dt \\ \dot{K}_t = F(K_t, A_t N_t (1 - l_t)) - \mu K_t - N_t c_t \\ N_t = N_0 e^{nt} \\ K_0 \text{ given} \end{cases}$$

Population N_t grows at a constant exogenous rate n. Each individual devotes a proportion l_t of his time endowment to leisure. A_t is technology or, more precisely, an exogenous index of labor efficiency. Nothing is assumed as yet regarding its evolution. Total labor, expressed in efficiency units, is $A_t N_t (1 - l_t)$. K_t is capital, which depreciates at rate μ. The production function has constant returns to scale and is well-behaved.

The instantaneous utility of the representative agent depends on his consumption per capita c and on leisure l. It is discounted[1] at rate ρ. It is well known, in particular since King-Plosser-Rebelo(1988), that strong restrictions have to be imposed upon the utility function in order to ensure existence of a balanced growth path when technical progress has a deterministic trend. We therefore retain the utility function

$$U(c,l) = \frac{(cv(l))^{1-1/\sigma}}{1-1/\sigma} \qquad \text{or} \qquad U(c,l) = lnc + lnv(l)$$

Note that our normalization differs slightly from the one in King-Plosser-Rebelo(1988), as function $v(l)$ appears under the exponent $1 - 1/\sigma$. This ensures a better continuity with the logarithmic case as σ tends to one. Function $v(l)$ is increasing and positive. We also assume that $U(c,l)$ is concave, which will be made precise later on.

The model yields the following optimality conditions :

$$U'_c(c,l) = c^{-1/\sigma} v(l)^{1-1/\sigma} = x$$

$$lv'(l)/v(l) = wl/c$$

$$w = AF'_2(K, AN(1-l))$$

$$r = F'_1(K, AN(1-l)) - \mu$$

$$\dot{x}/x = \rho + n - r$$

$$\dot{K} = F(K, AN(1-l)) - \mu K - Nc$$

w and r are the implicit wage and interest rates, whereas x is the shadow price of consumption. Note that the elasticity of function $v(l)$ has to equal the ratio between (implicit) leisure and consumption expenses. This condition derives from the standard optimality condition $U'_l(c,l) = c^{1-1/\sigma} v(l)^{-1/\sigma} v'(l) = xw$ when it is combined with the first condition.

Let us now define variables in intensive form :

$$\tilde{y} = y/A = Y/(AN) \quad \tilde{c} = c/A = C/(AN) \quad \tilde{k} = k/A = K/(AN)$$

Taking into account labor supply variability, we have

$$\tilde{y} = \frac{F(K, AN(1-l))}{AN} = (1-l) f\left(\frac{K}{AN(1-l)}\right) = (1-l) f\left(\frac{\tilde{k}}{1-l}\right)$$

Let $\tilde{x} = xA^{1/\sigma}$ and $\tilde{w} = w/A$. The model becomes :

$$\tilde{c}^{-1/\sigma} v(l)^{1-1/\sigma} = \tilde{x}$$

$$\tilde{c} v'(l)/v(l) = \tilde{w}$$

$$\widetilde{w} = f\left(\frac{\widetilde{k}}{1-l}\right) - \frac{\widetilde{k}}{1-l} f'\left(\frac{\widetilde{k}}{1-l}\right)$$

$$r = f'\left(\frac{\widetilde{k}}{1-l}\right) - \mu$$

$$\dot{\widetilde{x}}/\widetilde{x} = \rho + n - r + \frac{1}{\sigma}\frac{\dot{A}}{A}$$

$$\dot{\widetilde{k}} = (1-l) f\left(\widetilde{k}/(1-l)\right) - \left(\mu + n + \dot{A}/A\right)\widetilde{k} - \widetilde{c}$$

The model can be reduced to a differential system in the unknowns $\widetilde{k}$ and $\widetilde{x}$.

The balanced growth path

We now assume that the technical progress factor A_t grows at a constant rate γ. The system has a stationary point $\widetilde{k}^*$, $\widetilde{x}^*$ which depicts a balanced growth path where global quantities Y, K and C grow at the rate $g = n + \gamma$ while leisure is constant at a level l^*.

Let us define a modified discount rate $\rho' = \rho + \frac{1-\sigma}{\sigma}\gamma$. The interest rate is

$$r^* = \rho + n + \gamma/\sigma = \rho' + g = f'\left(\widetilde{k}^*/(1-l^*)\right) - \mu$$

Let α be the share of gross profits in national income, computed at the stationary point :

$$\alpha = \frac{\widetilde{k}^*}{1-l^*} \frac{f'\left(\widetilde{k}^*/(1-l^*)\right)}{f\left(\widetilde{k}^*/(1-l^*)\right)} = \frac{(r^* + \mu)\,\widetilde{k}^*}{\widetilde{y}^*}$$

Wages account for a proportion $1-\alpha$ of production :

$$\widetilde{w}^*(1-l^*) = (1-\alpha)\widetilde{y}^*$$

Let us define gross investment $I = \dot{K} + \mu K$ and its share in production $s = (I/Y)^*$. Then :

$$\left(\frac{I}{K}\right)^* = \mu + g \quad , \quad \left(\frac{K}{Y}\right)^* = \frac{\alpha}{\rho' + g + \mu} \quad , \quad \left(\frac{I}{Y}\right)^* = s = \frac{\alpha(g+\mu)}{\rho' + g + \mu}$$

Lastly, we let ξ denote the equilibrium ratio between leisure and consumption expenses. As we saw, it is equal to the elasticity of the utility function $v(l)$.

$$\xi = \frac{\tilde{w}^* l^*}{\tilde{c}^*} = \frac{l^* v'(l^*)}{v(l^*)} = \frac{l^*}{1-l^*}\frac{1-\alpha}{1-s}$$

The linearized model

Let us first comment on the linearization and, more generally, the comparative dynamics exercises that we shall perform. We shall show how a simple variational calculus can be used in practice to derive precise and quantitative results. Some technical conditions obviously must be met to make this kind of analysis valid. We won't delve into this matter but we may indicate some references which provide the formal basis for this methodology. From a mathematical standpoint, the issue is to examine the dependence of the solution of a system of differential equations on parameters or on initial values : see for instance Pontryagin(1962). Oniki(1973) provides, on these lines, a general analysis ot the comparative dynamics of optimal control problems. A special feature of our framework however is that we want to consider the influence of time-varying parameters such as the level of technical progress A_t in the above problem, or the level of wages and interest rate in the consumer problem which we shall consider below. Formal methods exist to extend the analysis to such a case : see, for instance, Judd(1987). A related technique, introduced in economics by Wan(1970) and recently used by Epstein(19XX), is the so-called Volterra derivatives of functionals. It describes for instance the influence on total utility of a small change in consumption, during an infinitesimal time-interval. Again, we shall not make explicit this connection.

Let us rather spell the basic notations and rules that we use. For any variable z, δz_t denotes its variation at date t, that is the (infinitesimal) difference between z_t and its benchmark value z_t^*. $\delta z_t/z_t$ denotes the relative variation $\delta ln z_t$. These relative variations are the endogenous variables in the loglinearized differential system. They have then to be considered as "indivisible" expressions. But they may also be considered as ratios since $\delta z_t/z_t = \delta ln z_t = \delta z_t/z_t^*$. The notation therefore does not create any risk of error.

An important rule is that the order of application of operator δ and of the time-derivative operator d/dt can be inversed. The same notation $\delta \dot{z}_t$ therefore denotes $\delta(\dot{z}_t)$ as well as $d/dt\,(\delta z_t)$. Similarily $\delta \dot{z}_t/z_t$ denotes $\delta(\dot{z}_t/z_t)$ as well as $d/dt\,(\delta z_t/z_t)$. All standard rules of differential calculus apply, and we shall even be led to integrate these variations over a time interval.

We linearize the model around the balanced growth path. We shall analyze the effects of infinitesimal variations of initial capital K_0 and of the trajectory of the technological factor A_t. All other parameters are unchanged and so is the exogenous evolution of population N_t.

Two new parameters intervene in this linearization. σ_f is the elasticity of substitution between capital and labor, while η is the opposite of the elasticity of function $v'(l)/v(l)$. Both of them are calculated at the stationary point. We thus use σ_f and α to parameterize the production function and ξ, elasticity of function $v(l)$, and η to parameterize the utility of leisure. These four parameters may be treated independently.

Linearization of the static part of the consumer's behavior yields

$$\begin{pmatrix} \delta\tilde{c}/\tilde{c} \\ \delta l/l \end{pmatrix} = \mathcal{N} \begin{pmatrix} \delta\tilde{w}/\tilde{w} \\ \delta\tilde{x}/\tilde{x} \end{pmatrix}$$

with

$$\mathcal{N} = \begin{pmatrix} n_{cw} & n_{cx} \\ n_{lw} & n_{lx} \end{pmatrix} = \frac{1}{\eta + (1-\sigma)\xi} \begin{pmatrix} (1-\sigma)\xi & -\sigma\eta \\ -1 & -\sigma \end{pmatrix}$$

It can be checked, from homogeneity considerations, that matrix $\mathcal{N}$ also describes the influence of the undeflated variations $\delta w/w$ and $\delta x/x$ on undeflated consumption $\delta c/c$ and leisure $\delta l/l$. In particular, n_{cw} and n_{lw} are the so-called x-constant, or Frischian, wage elasticities of consumption and leisure which are often considered in the literature[2].

The second static relationship is the short run labor demand. Let us define, for convenience, work hours $h = 1 - l$. Then

$$\frac{\delta h}{h} = -\frac{l^*}{1-l^*}\frac{\delta l}{l} = -\frac{(1-s)\xi}{1-\alpha}\frac{\delta l}{l}$$

We obtain

$$\frac{\delta\tilde{w}}{\tilde{w}} = \frac{\alpha}{\sigma_f}\frac{\delta\left(\tilde{k}/h\right)}{\tilde{k}/h} = \frac{\alpha}{\sigma_f}\left(\frac{\delta\tilde{k}}{\tilde{k}} + \frac{(1-s)\xi}{1-\alpha}\frac{\delta l}{l}\right)$$

Transposed in undeflated variables, labor demand becomes

$$\frac{\delta w}{w} = \frac{\alpha}{\sigma_f}\left(\frac{\delta K}{K} + \frac{(1-s)\xi}{1-\alpha}\frac{\delta l}{l}\right) + \left(1 - \frac{\alpha}{\sigma_f}\right)\frac{\delta A}{A}$$

The wage-elasticity of labor demand is equal to σ_f/α. A positive technological shock shifts upwards the demand for labor if and only if capital-labor elasticity σ_f is higher than the share of profits α. In the opposite case of a very low elasticity of substitution, the increase in labor efficiency would induce the firm to hire less labor, at a given wage rate.

Elimination of the wage rate yields the following relationship.

$$\begin{pmatrix} \delta\widetilde{c}/\widetilde{c} \\ \delta l/l \end{pmatrix} = \mathcal{M} \begin{pmatrix} \delta\widetilde{k}/\widetilde{k} \\ \delta\widetilde{x}/\widetilde{x} \end{pmatrix}$$

with

$$\mathcal{M} = \begin{pmatrix} m_{ck} & m_{cx} \\ m_{lk} & m_{lx} \end{pmatrix} = \frac{1}{\eta + (1-\sigma)\xi + \frac{\alpha(1-s)\xi}{(1-\alpha)\sigma_f}} \begin{pmatrix} (1-\sigma)\xi\dfrac{\alpha}{\sigma_f} & -\sigma\left(\eta + \dfrac{\alpha(1-s)\xi}{(1-\alpha)\sigma_f}\right) \\ -\dfrac{\alpha}{\sigma_f} & -\sigma \end{pmatrix}$$

Taking into account the dynamic equations we are led to the following system of linear differential equations

$$\begin{pmatrix} \delta\dot{\widetilde{k}}/\widetilde{k} \\ \delta\dot{\widetilde{x}}/\widetilde{x} \end{pmatrix} = J \begin{pmatrix} \delta\widetilde{k}/\widetilde{k} \\ \delta\widetilde{x}/\widetilde{x} \end{pmatrix} + \begin{pmatrix} -1 \\ 1/\sigma \end{pmatrix} \left(\delta\dot{A}/A\right)$$

with the jacobian matrix

$$J = \begin{pmatrix} \rho' - (1-s)\dfrac{\rho'+g+\mu}{\alpha}(m_{ck} + \xi m_{lk}) & -(1-s)\dfrac{\rho'+g+\mu}{\alpha}(m_{cx} + \xi m_{lx}) \\ (1-\alpha)\dfrac{\rho'+g+\mu}{\sigma_f}\left(1 + \dfrac{(1-s)\xi}{1-\alpha} m_{lk}\right) & (1-s)\xi\dfrac{\rho'+g+\mu}{\sigma_f} m_{lx} \end{pmatrix}$$

Note that this system holds for any trajectory of (small) variations of A_t around its trend. Formally, $\delta\dot{A}_t/A_t$ acts as a forcing variable in the dynamical system which determines the evolution of $\delta\widetilde{k}/\widetilde{k}$ and $\delta\widetilde{x}/\widetilde{x}$.

We first examine the intrinsic dynamics of the model.

Proposition :

If the two conditions

$$\rho' = \rho + \frac{1-\sigma}{\sigma}\gamma > 0 \quad , \quad \eta + (1-\sigma)\xi \geq 0$$

hold, matrix J has two real eigenvalues of opposite signs. The dynamics is therefore of the saddlepoint type.

The first condition is a well-known necessary condition for the existence of a solution to the maximization problem of the representative agent. It ensures that the agent cannot reach an infinite utility level.

The second condition expresses the concavity of function $U(c, l)$. Note that η may be negative if $\sigma < 1$. This condition implies the concavity of function $v(l)^{1-1/\sigma}$, which can be seen to be $\sigma\eta + (1-\sigma)\xi \geq 0$, but it is stronger.

This analysis covers several particular cases. The Cass(1965) model, with exogenous labor supply, is obtained for $\xi = 0$. The logarithmic case is obtained for $\sigma = 1$ and yields an x-constant elasticity of leisure equal to $n_{lw} = -1/\eta$. Another polar case is $\eta = 0$. As η is the elasticity of function $v'(l)/v(l)$, it is zero if this function is equal to a constant, say b. Then $v(l) = exp(bl)$ which implies $\xi = bl^*$ and $n_{lw} = -1/((1-\sigma)bl^*)$. An interesting special case is the one where simultaneously $\eta = 0$ and $\sigma = 1$. The utility function is $U(c, l) = lnc + lnv(l) = lnc + bl$ and n_{lw} becomes infinite. This case where the utility function is linear with respect to leisure has been considered by Hansen(1985) and Rogerson(1988), as the reduced form of a model where work hours are fixed and where the labor contract takes the form of a lottery.

The solution of the linearized model, with a technological shock

We now retain a specific form for the technological shock, with a constant adjusment speed λ.

$$\frac{\delta \dot{A}_t}{A_t} = -\lambda \frac{\delta A_t}{A_t} \qquad \text{i.e.} \qquad \frac{\delta A_t}{A_t} = e^{-\lambda t} \frac{\delta A_0}{A_0}$$

The model is solved in the usual way, eliminating the explosive component and adding a particular solution to take care of the forcing factor $\delta \dot{A}/A$.

Let θ be the absolute value of the negative eigenvalue and v the ratio between the second and the first components of the associated eigenvector. v is negative.

Proposition :

The solution of the linearized model is

$$\delta \tilde{k}_t / \tilde{k}_t = e^{-\theta t} \delta \tilde{k}_0 / \tilde{k}_0 + b_k \frac{e^{-\theta t} - e^{-\lambda t}}{\lambda - \theta} \delta A_0 / A_0$$

$$\delta \tilde{x}_t / \tilde{x}_t = v \delta \tilde{k}_t / \tilde{k}_t + b_x \delta A_t / A_t$$

where

$$b_x = \frac{\lambda}{\lambda + \rho' + \theta} \left(v + \frac{1}{\sigma} \right) \quad , \quad b_k = \lambda + J_{kx} b_x$$

Note that we have chosen an expression for $\delta \tilde{k}_t / \tilde{k}_t$ which makes more transparent the continuity with the special case $\lambda = \theta$, where the solution is

$$\delta \tilde{k}_t / \tilde{k}_t = e^{-\theta t} \delta \tilde{k}_0 / \tilde{k}_0 + b_k t e^{-\theta t} \delta A_0 / A_0$$

$\delta\widetilde{k}_t/\widetilde{k}_t$ and $\delta A_t/A_t$ appear as the state variables of the model. All variables can be expressed in terms of the current values of these two variables. The value of the shadow price $\delta\widetilde{x}_t/\widetilde{x}_t$ is given in the proposition. The value of the undeflated shadow price follows :

$$\delta x_t/x_t = \delta\widetilde{x}_t/\widetilde{x}_t - (1/\sigma)\delta A_t/A_t = v\delta\widetilde{k}_t/\widetilde{k}_t + (b_x - 1/\sigma)\,\delta A_t/A_t$$

Similar expressions are obtained for the main variables :

$$\begin{pmatrix} \delta C_t/C_t \\ \delta I_t/I_t \\ \delta Y_t/Y_t \\ \delta l_t/l_t \\ \delta w_t/w_t \\ \delta r_t/r_t \end{pmatrix} = T \begin{pmatrix} \delta\widetilde{k}_t/\widetilde{k}_t \\ \delta A_t/A_t \end{pmatrix}$$

where matrix T is the following

$$\begin{pmatrix} m_{ck} + m_{cx}v & 1 + m_{cx}b_x \\ \dfrac{g+\mu-\theta}{g+\mu} & 1 + \dfrac{J_{kx}b_x}{g+\mu} \\ \alpha - (1-s)\xi\,(m_{lk} + m_{lx}v) & 1 - (1-s)\xi m_{lx}b_x \\ m_{lk} + m_{lx}v & m_{lx}b_x \\ \dfrac{\alpha}{\sigma_f}\left(1 + \dfrac{(1-s)\xi}{1-\alpha}\,(m_{lk} + m_{lx}v)\right) & 1 + \dfrac{\alpha}{\sigma_f}\dfrac{(1-s)\xi}{1-\alpha}m_{lx}b_x \\ -\dfrac{\rho'+g+\mu}{\rho'+g}\dfrac{1-\alpha}{\sigma_f}\left(1 + \dfrac{(1-s)\xi}{1-\alpha}\,(m_{lk} + m_{lx}v)\right) & -\dfrac{\rho'+g+\mu}{\rho'+g}\dfrac{1-\alpha}{\sigma_f}\dfrac{(1-s)\xi}{1-\alpha}m_{lx}b_x \end{pmatrix}$$

These relationships satisfy various consistency conditions. The factor price frontier is

$$\alpha\frac{r^*}{r^*+\mu}\frac{\delta r}{r} + (1-\alpha)\frac{\delta w}{w} = (1-\alpha)\frac{\delta A}{A}$$

It implies

$$\alpha\frac{r^*}{r^*+\mu}T_{rk} + (1-\alpha)T_{wk} = 0 \quad , \quad \alpha\frac{r^*}{r^*+\mu}T_{ra} + (1-\alpha)T_{wa} = (1-\alpha)$$

Similarily, the national income identity $Y = C + I$ implies

$$T_{yk} = (1-s)T_{ck} + sT_{ik} \quad , \quad T_{ya} = (1-s)T_{ca} + sT_{ia}$$

We deduce from these formulas the response functions, that is the trajectories followed by various variables after a technological shock. If we start from the stationary point, $\delta K_0/K_0 = 0$, and therefore $\delta\tilde{k}_0/\tilde{k}_0 = -\delta A_0/A_0$. Let us consider for instance consumption. We obtain

$$\delta C_t/C_t = T_{ck}\delta\tilde{k}_t/\tilde{k}_t + T_{ca}\delta A_t/A_t =$$

$$= T_{ck}\left(-e^{-\theta t} + b_k\frac{e^{-\theta t} - e^{-\lambda t}}{\lambda - \theta}\right)\frac{\delta A_0}{A_0} + T_{ca}e^{-\lambda t}\frac{\delta A_0}{A_0}$$

The response function appears as a weighted sum of two exponentials. It is therefore either monotonous or unimodal.

Impact effects in particular can be calculated :

$$\frac{\delta x_0}{x_0} = -\frac{\rho' + \theta}{\lambda + \rho' + \theta}\left(v + \frac{1}{\sigma}\right)\frac{\delta A_0}{A_0}$$

$$\frac{\delta C_0}{C_0} = -m_{cx}\left(\frac{\rho' + \theta}{\lambda + \rho' + \theta}\left(v + \frac{1}{\sigma}\right) + \frac{1 - \sigma}{\sigma}\frac{\xi\left(\sigma_f - \alpha\right)}{\eta\sigma_f + \frac{(1-s)\xi}{1-\alpha}\alpha}\right)\frac{\delta A_0}{A_0}$$

$$\frac{\delta l_0}{l_0} = -m_{lx}\left(\frac{\rho' + \theta}{\lambda + \rho' + \theta}\left(v + \frac{1}{\sigma}\right) + \frac{\alpha}{\sigma\sigma_f} - \frac{1}{\sigma}\right)\frac{\delta A_0}{A_0}$$

As v is negative, the signs of all three expressions are ambiguous. Note that the influence of the adjustment speed λ is totally explicit as all other parameters in these formulas are independent of λ. Among these parameters are the eigenvalue θ and the associated coefficient v, which are explicit but cumbersome functions of the basic parameters. A precise theoretical analysis of the influence of these basic parameters is therefore difficult. Fortunately, the polar cases of permanent ($\lambda = 0$) or purely transitory ($\lambda = +\infty$) shocks yield simple and explicit results.

Impact effects of permanent shocks

$\lambda = 0$ implies $b_k = b_x = 0$, $T_{ca} = T_{ia} = T_{ya} = T_{wa} = 1$ and $T_{la} = T_{ra} = 0$. The core of the model becomes

$$\delta\tilde{k}_t/\tilde{k}_t = e^{-\theta t}\delta\tilde{k}_0/\tilde{k}_0 \quad , \quad \delta\tilde{x}_t/\tilde{x}_t = v\delta\tilde{k}_t/\tilde{k}_t$$

All variables adjust monotonically to their long run values, with an adjustment speed θ.

After some tedious calculation, the impact effects can be signed.

Proposition

In the case of a permanent shock,

$\delta C_0/C_0$ *is positive iff*

$$\left(\eta+\frac{(1-s)\xi}{1-\alpha}\frac{\alpha}{\sigma_f}\right)\left(\sigma_f-\sigma\alpha\right)+\xi(1-\sigma)\left(\sigma_f-\alpha\right)>0$$

and

$$\eta+\frac{(1-s)\xi}{1-\alpha}\frac{\alpha}{\sigma_f}+\xi\left(1-\frac{\alpha}{\sigma_f}\right)>0$$

$\delta l_0/l_0$ *is negative iff*

$$(1-s)\left(\sigma_f-\alpha\right)+(1-\alpha)\sigma_f(\sigma-1)>0$$

$\delta x_0/x_0$ *is negative iff*

$$(1-s)\xi\sigma\left(\sigma_f-\alpha\right)^2+(1-\alpha)\sigma_f\left(\sigma_f-\sigma\alpha\right)\left(\eta+(1-\sigma)\xi+\frac{(1-s)\xi}{1-\alpha}\frac{\alpha}{\sigma_f}\right)>0$$

and

$$\eta+\frac{(1-s)\xi}{1-\alpha}\frac{\alpha}{\sigma_f}+\xi\left(1-\frac{\sigma\alpha}{\sigma_f}\right)>0$$

In practice, the second condition which appears in the signing of $\delta C_0/C_0$ or $\delta x_0/x_0$ only plays a secondary role. It is usually satisfied as soon as the first condition is.

Simple inspection of these conditions shows that values of σ_f and σ close to unity are sufficient to imply a positive response of consumption and labor (that is a negative response of leisure), as well as a negative response of the shadow price of consumption. A technological shock is therefore likely to have usual effects. The picture however may change if we enlarge the admissible range of variation of the parameters.

We first note that the ambiguity is present even if labor supply is exogenous, namely if $\xi=0$:

In the case of exogenous labor supply, and of a permanent shock, $\delta C_0/C_0$ *is positive and* $\delta x_0/x_0$ *negative iff* $\sigma_f>\sigma\alpha$.

The general case is illustrated in figures 1 and 2, which offer two different views of the parameter space[3] . It is seen that a variety of results are possible if we move away from the benchmark case $\sigma = \sigma_f = 1$. Curves CC, xx and ll are respectively the loci of zero initial effect on consumption, shadow price and leisure. They intersect in figure 1 at the point $\sigma = 1$, $\sigma_f = \alpha$, where curves CC and xx admit the straight line $\sigma_f = \alpha\sigma$ as a common tangent.

Figures 1 et 2

The case of purely transitory shocks

We now consider the second polar case where $\lambda = +\infty$ and shocks have no persistence at all. This is only a limit-case. In a continuous time model, an instantaneous shock changes the current values of production, consumption and investment. However this change in the level of flows, which only lasts for an infinitesimal time interval, has no influence on the later evolution of the economy. It is nevertheless useful to study this case as it gives the limit of the effects which are observed when the persistence of the shocks goes to zero.

When λ tends to infinity, b_x tends to $v + 1/\sigma$ and $b_k/(\lambda - \theta)$ to 1. It follows that $\delta x_0/x_0 = 0$ and $\delta\widetilde{k}_t/\widetilde{k}_t = \left(-e^{-\theta t} + e^{-\lambda t} + e^{-\theta t}\right)\delta A_0/A_0 = 0$.

The evolution of $\widetilde{k}_t$, for $t > 0$, is unchanged. As A_t does not change either, no variable changes after the instantaneous shock.

The main impact effects are the following.

Proposition

In the case of an instantaneous shock,

$$\frac{\delta x_0}{x_0} = 0$$

$$\frac{\delta C_0}{C_0} = \frac{(1-\sigma)\xi\left(\sigma_f - \alpha\right)}{\sigma_f\left(\eta + (1-\sigma)\xi + \frac{(1-s)\xi}{1-\alpha}\frac{\alpha}{\sigma_f}\right)}\frac{\delta A_0}{A_0}$$

$$\frac{\delta l_0}{l_0} = \frac{-\left(\sigma_f - \alpha\right)}{\sigma_f\left(\eta + (1-\sigma)\xi + \frac{(1-s)\xi}{1-\alpha}\frac{\alpha}{\sigma_f}\right)}\frac{\delta A_0}{A_0}$$

Response functions in the general case

Although we have obtained explicit expressions of the response functions, they are too complicated to allow an analysis as precise as the one we have just made of permanent or instantaneous shocks.

Let us therefore begin with a numerical analysis. For comparison, we retain the parameterization of King-Plosser-Rebelo(1988). It is calibrated on the US economy, for quarterly datas.

ρ	μ	n	γ	α	σ_f	σ	ξ	η	λ
0.012	0.025	0	0.004	0.42	1	1	3.30	10	0.1

The value of ξ is chosen so as to obtain a proportion of work hours $1 - l^*$ equal to 0.2. This value is equal to $4(1-s)/(1-\alpha)$. The elasticity η is then chosen to obtain a wage elasticity of labor supply equal to 0.4, namely $(l^*/(1-l^*))\, n_{lw} = (l^*/(1-l^*))/\eta = 0.4$.

The balanced growth path is described in the following table.

r	s	l^*	K/Y^*
0.016	0.297	0.8	10.2

The stable eigenvalue is θ =0.0386. If the shock is permanent, half of the adjustment of all variables to their long run levels requires $(ln2)/\theta = 18$ quarters.

Figure 3

Figure 3 reports the response functions. We plot simultaneously the responses to an instantaneous shock ($\lambda = \infty$), to a permanent shock ($\lambda = 0$) and to a shock with the realistic adjustment speed $\lambda = 0.1$. The response functions to an instantaneous shock reduce to vertical bars. The response functions to a permanent shock are recognized at their monotonous shape, and at the permanent effects on C, I and Y. The responses to a shock with an intermediate adjustment speed of 10% have the usual shape and we note, in particular, the hump-shape response of consumption.

It is also possible to quantify the co-movements of the variables. In the usual setting of Real Business Cycle models, shocks are stochastic and the model is written in discrete time. One can use the response functions, that is the Moving Average form of the stochastic processes followed by the variables, to calculate their covariances. We have here a deterministic model written in continuous time. By analogy, we define in the following way the "covariance" between two variables :

$$Cov(C,Y) = \int_0^{+\infty} \frac{\delta C_t}{C_t}\frac{\delta Y_t}{Y_t}\, dt$$

As $\delta C_t/C_t$ and $\delta Y_t/Y_t$ are combinations of two exponentials, it is easy to calculate this quantity :

$$Cov(C,Y) = (1+b_k)^2 \frac{T_{ck}T_{yk}}{2\theta} - (1+b_k)\frac{T_{ck}\left(b_k T_{yk} + T_{ya}\right) + T_{yk}\left(b_k T_{ck} + T_{ca}\right)}{\theta + \lambda} +$$

$$+\frac{(b_k T_{ck}+T_{ca})\left(b_k T_{yk}+T_{ya}\right)}{2\lambda}$$

Using our benchmark calibration, we present in the following table the standard-deviations of the main variables, as a proportion of the standard-deviation of output, and the correlations with output. The numerical values duplicate, with very good precision, the results in King-Plosser-Rebelo(1988, table 4, line 6). It thus appears that using a continuous time deterministic model does not affect the quantitative properties of the model. The ultimate reason for this is that the solution to real business cycle models does not depart very much from certainty-equivalents, which is what legitimates in the first place the solution method of King-Plosser-Rebelo(1988).

	C	I	$1-l$	w	r
$SD./SD_Y$	0.69	2.17	0.16	0.89	1.75
$Corr(.,Y)$	0.85	0.92	0.75	0.99	0.46

We now study the influence of the degree of persistence of the technological shocks. Inspection of the formulas giving the impact effects of the shocks first shows that changes in λ do not change the sign of $\delta x_0/x_0$, which only depends on the sign of $v+1/\sigma$. An increase in the persistence of the shock, that is a decrease of λ, always increases the size of the initial response of x_0. This is rather intuitive. As we shall see, x_0 is the channel through which operate all intertemporal effects. Whatever their sign, the stronger the persistence, the stronger are these effects. A change in λ also affects $\delta C_0/C_0$ and $\delta l_0/l_0$, in a direction which also depends on the sign of $v+1/\sigma$. Contrary to what happens to $\delta x_0/x_0$, it may very well result in a change of the sign of these two initial effects.

These remarks allow us to understand how changes in λ affect the regioning in figure 1 and 2. Consider for instance figure 1. Curve xx and the point where the three curves intersect are unaffected. An increase in λ implies a decrease in $\delta C_0/C_0$ and in $\delta l_0/l_0$ when $v+1/\sigma$ is positive, that is in the region where $\delta x_0/x_0$ is negative. It follows that the curve CC shifts upward, while remaining tangent to the curvexx. Curve llrotates counter-clock-wise around the intesection point. Figure 4 illustrate these shifts when λ increases from 0 to 0.1.

Figure 4

We may also examine the influence of λ in our benchmark case. Then $v+1/\sigma$ is positive and $\delta x_0/x_0$ is negative. As $\sigma=\sigma_f=1$, we have

$$\frac{\delta C_0}{C_0}=-m_{cx}\frac{\rho'+\theta}{\lambda+\rho'+\theta}\left(v+\frac{1}{\sigma}\right)\frac{\delta A_0}{A_0}$$

$$\frac{\delta l_0}{l_0}=\left(-m_{lx}\frac{\rho'+\theta}{\lambda+\rho'+\theta}\left(v+\frac{1}{\sigma}\right)+m_{lx}(1-\alpha)\right)\frac{\delta A_0}{A_0}$$

$\delta C_0/C_0$ is positive. $\delta l_0/l_0$ is the sum of a positive and a negative term. It turns out however that the negative term dominates for all values of λ. A decrease in λ therefore increases $\delta C_0/C_0$ and $\delta l_0/l_0$, but this means that it reduces the absolute value of $\delta l_0/l_0$ In other words, an increase in the persistence of shocks amplifies the response of consumption, but reduces the response of labor. This result is apparent on figure 3. The economic reason for it is that the dominant effect of a shock on leisure or labor is the effect which operates through current wages. This effect is independent of the degree of persistence. On the other hand, an increase in persistence amplifies the intertemporal effects which operate through x_0. As it amplifies effects which are dominated, it reduces the overall response of leisure or labor.

Lastly, we examine the influence of λ on the path followed by capital intensity $\tilde{k}$, the state variable of the model. The issue is to assess the validity of the insight according to which more persistence of the shock means a slower adjustment of all variables to their long run levels. The evolution of $\tilde{k}$ is described by the following equation :

$$\frac{\delta \tilde{k}_t}{\tilde{k}_t} = \left(b_k \frac{e^{-\theta t} - e^{-\lambda t}}{\lambda - \theta} - e^{-\theta t} \right) \frac{\delta A_0}{A_0}$$

When $\lambda = 0$, $\delta \tilde{k}_t / \tilde{k}_t$ decreases monotonously to zero from its initial level $-\delta A_0/A_0$. For all positive values of λ, the path followed by $\delta \tilde{k}_t / \tilde{k}_t$ is characterized by an overshooting, as $\delta \tilde{k}_t / \tilde{k}_t$ increases from its negative initial value and becomes positive before decreasing back to zero. An increase in λ implies an increase in $\delta \tilde{k}_t / \tilde{k}_t$ for small values of t, but a decrease for high values of t. These evolutions are pictured on figure 5. One might say that a decrease in the persistence of shocks (a decrease in λ) implies a faster adjustment. This however amounts to neglecting the overshooting phenomenon. It is important to recognize that the overall amount of overshooting is the largest for intermediate levels of λ. We shall later see that this pattern of overshooting, also characterizes the interest rate, and this will play an important role in the dynamics of the model.

Figure 5

The value-function

It is possible to calculate the effects of variations of initial values of capital and technology on the utility level of the representative agent. In other words, we study the value-function $V(K, A)$ of the representative agent.

Using the optimality conditions, we find

$$\delta V_0 = \frac{\sigma - 1}{\sigma} \rho' V_0^{*} \int_0^{+\infty} e^{-\rho' t} \left(\frac{\delta c_t}{c_t} + \xi \frac{\delta l_t}{l_t} \right) \, dt$$

which yields, after integration,

$$\delta V_0 = \frac{\sigma - 1}{\sigma} \rho' V_0^{*} \left(\frac{\alpha}{(1-s)(\rho' + g + \mu)} \frac{\delta K_0}{K_0} + \frac{1 - \alpha}{(1-s)(\rho' + \lambda)} \frac{\delta A_0}{A_0} \right)$$

As $V_0^* = U\left(c_0^*, l^*\right)/\rho'$ has the sign of $(\sigma-1)/\sigma$, we check that a positive δK_0 or a positive δA_0 implies a positive δV_0. More capital, or a better technology, obviously means higher welfare.

The labor market and a first economic interpretation

The previous analysis has determined the effects of a technological shock. We now look for a precise economic interpretation of these effects.

As a preliminary step, we may consider the static part of the model, and in particular labor market equilibrium. Using work hours $h = 1 - l$ rather than leisure, initial effects are desribed by the following relationships.

$$\frac{\delta C_0}{C_0} = n_{cw}\frac{\delta w_0}{w_0} + n_{cx}\frac{\delta x_0}{x_0}$$

$$\frac{\delta h_0}{h_0} = -\frac{(1-s)\xi}{1-\alpha}\left(n_{lw}\frac{\delta w_0}{w_0} + n_{lx}\frac{\delta x_0}{x_0}\right)$$

$$\frac{\delta w_0}{w_0} = -\frac{\alpha}{\sigma_f}\frac{\delta h_0}{h_0} + \left(1 - \frac{\alpha}{\sigma_f}\right)\frac{\delta A_0}{A_0}$$

The last two equations describe labor market equilibrium. A technological shock has a direct influence on labor demand. If we make the plausible assumption that the factorial elasticity of substitution is higher than the share of profits, a positive technological shock shifts labor demand upwards. The shock also modifies the shadow price of consumption, and therefore shifts the labor supply curve. This shift sums up all intertemporal effects which affects the consumer through his expectations. In the usual case, a positive technological shock induces the agent to take more leisure and it shifts upwards the labor supply curve. The overall effect on labor remains a priori ambiguous. It is usually positive as the labor-demand effects dominates. Figure 6 represents this case. It is clear however that a variety of results are possible. The figure can also be used to describe the effects of an instantaneous shock. In such a case x_0 is unchanged and the shock only affects the demand curve.

Figure 6

The effect on consumption follows. A decrease in x_0 induces more consumption. If $\sigma < 1$, the wage increase induces to substitute consumption for labor, which reinforces the positive effect on consumption. If $\sigma > 1$, it runs counter to the effect of the shadow price.

Thus a static analysis shows that all initial effects result from a direct effect of $\delta A_0/A_0$ and an indirect effect which operates through the variation $\delta x_0/x_0$ of the shadow price. The deeper issue of course is to interpret the mechanisms governing the variation of the shadow price.

Following the early analysis of King-Plosser-Rebelo(1988) it is now common to distinguish between three broad kinds of effects : an overall wealth effect, a contemporaneous effect linked to the change in current wages and a set of intertemporal substitution effects : see, among many others, Hairault(1995). Our aim is to make more precise this classification and to quantify its elements. It seems to us rather clear that a precise interpretation of the effects of a technological shock has to rely on the traditional distinction between substitution and income effects. As this distinction a priori concerns the behavior of a pure consumer, facing given prices, we must leave the optimal growth version of the model and focus instead upon the decentralized equilibrium version.

Substitution and income effects in the consumer's behavior

The representative consumer faces on his infinite horizon a sequence of wages $\{w_t\}$ and interest rates $\{r_t\}$. In order to identify substitution and income effects, we need however to focus on intertemporal prices, rather than on interest rates. We therefore define discounted prices $q_t = e^{-R_t}$, where $R_t = \int_0^{+\infty} r_s \, ds$.

Let B_0 be the consumer's initial level of financial asset and D_t the dividends received at date t. The consumer's intertemporal budget constraint is

$$\int_0^{+\infty} e^{-R_t} C_t \, dt = B_0 + \int_0^{\infty} e^{-R_t} \left(w_t N_t (1 - l_t) + D_t \right) \, dt =$$

$$= K_0 + \int_0^{\infty} e^{-R_t} w_t N_t (1 - l_t) \, dt$$

or equivalently

$$\int_0^{\infty} q_t N_t c_t \, dt = K_0 + \int_0^{\infty} q_t w_t N_t (1 - l_t) \, dt$$

The ultimate level of the consumer's wealth appears as the sum of the value K_0 of the stock of capital and the discounted value of future wages. A technological shock does not affect K_0. The only wealth effects caused by a technological shock therefore transit through the variations of wages or intertemporal prices.

The other relationships which characterize the consumers'behavior have already been described. The shadow price of consumption varies according to $\dot{x}_t / x_t = \rho - r_t + n$. It follows that $x_t = x_0 e^{(\rho+n)t} q_t$, and therefore

$$\delta x_t / x_t = \delta x_0 / x_0 + \delta q_t / q_t$$

Consumption and leisure are such that

$$\begin{pmatrix} \delta c_t/c_t \\ \delta l_t/l_t \end{pmatrix} = \mathcal{N} \begin{pmatrix} \delta w_t/w_t \\ \delta x_t/x_t \end{pmatrix} = \mathcal{N} \begin{pmatrix} \delta w_t/w_t \\ \delta x_0/x_0 + \delta q_t/q_t \end{pmatrix}$$

The initial value $\delta x_0/x_0$ may be determined using the intertemporal budget constraint, which we now linearize.

The benchmark balanced growth path is such that

$$r^* = \rho' + g \quad , \quad q_t^* = e^{-(\rho'+g)t} \quad , \quad Y_t^* = Y_0^* e^{gt}$$

$$\left(\frac{C_t}{Y_t}\right)^* = 1 - s \quad , \quad \left(\frac{w_t N_t (1-l_t)}{Y_t}\right)^* = 1 - \alpha \quad , \quad \left(\frac{K_t}{Y_t}\right)^* = \frac{\alpha}{\rho' + g + \mu} = \frac{\alpha - s}{\rho'}$$

The linearization of the budget constraint yields

$$\int_0^\infty q_t^* N_t c^* \left(\frac{\delta q_t}{q_t} + \frac{\delta c_t}{c_t}\right) dt = K_0^* \frac{\delta K_0}{K_0} + \int_0^\infty q_t^* w^* N_t (1 - l_t^*) \left(\frac{\delta q_t}{q_t} + \frac{\delta w_t}{w_t} - \frac{(1-s)\xi}{1-\alpha} \frac{\delta l_t}{l_t}\right) dt$$

or

$$(1-s) \int_0^\infty e^{-\rho' t} \left(\frac{\delta q_t}{q_t} + \frac{\delta c_t}{c_t}\right) dt = \frac{\alpha - s}{\rho'} \frac{\delta K_0}{K_0} + (1-\alpha) \int_0^\infty e^{-\rho' t} \left(\frac{\delta q_t}{q_t} + \frac{\delta w_t}{w_t} - \frac{(1-s)\xi}{1-\alpha} \frac{\delta l_t}{l_t}\right) dt$$

Substituting $\delta c_t/c_t$ and $\delta l_t/l_t$ by their expressions, we may calculate $\delta x_0/x_0$ to obtain the following.

Proposition

The variations of the consumer's demand functions may be decomposed in the following way :

$$\frac{\delta c_t}{c_t} = \frac{(1-\sigma)\xi}{\eta + (1-\sigma)\xi} \frac{\delta w_t}{w_t} - \frac{\eta\sigma}{\eta + (1-\sigma)\xi} \frac{\delta q_t}{q_t} +$$

(contemporaneous substitution effects)

$$+ \frac{\eta(\alpha - s)}{(1-s)(\eta + \xi)} \left[\frac{\delta K_0}{K_0} + \rho' \int_0^\infty e^{-\rho' u} \left(-\frac{\delta q_u}{q_u} + \frac{1-\alpha}{\alpha - s} \frac{\delta w_u}{w_u}\right) du\right] +$$

(intertemporal income effects)

$$+ \frac{\eta\sigma}{\eta + (1-\sigma)\xi} \rho' \int_0^\infty e^{-\rho' u} \left(\frac{\delta q_u}{q_u} + \frac{\xi}{\eta + \xi} \frac{\delta w_u}{w_u}\right) du$$

(intertemporal substitution effects)

as well as

$$\frac{\delta l_t}{l_t} = -\frac{1}{\eta + (1-\sigma)\xi}\frac{\delta w_t}{w_t} - \frac{\sigma}{\eta + (1-\sigma)\xi}\frac{\delta q_t}{q_t} +$$

(contemporaneous substitution effects)

$$+\frac{\alpha - s}{(1-s)(\eta + \xi)}\left[\frac{\delta K_0}{K_0} + \rho' \int_0^\infty e^{-\rho' u}\left(-\frac{\delta q_u}{q_u} + \frac{1-\alpha}{\alpha - s}\frac{\delta w_u}{w_u}\right) du\right] +$$

(intertemporal income effects)

$$+\frac{\sigma}{\eta + (1-\sigma)\xi}\rho' \int_0^\infty e^{-\rho' u}\left(\frac{\delta q_u}{q_u} + \frac{\xi}{\eta + \xi}\frac{\delta w_u}{w_u}\right) du$$

(intertemporal substitution effects)

The same formulas, taken at time $t = 0$, yield initial effects. One should note however that, in such a case, $q_0 = 1$ and therefore $\delta q_0 / q_0 = 0$.

This decomposition follows from standard microeconomic principles. In the present setting, some comments however are in order.

Contemporaneous effects describe the influence of changes in wage or price on the demands expressed at the same date. They are x_0-constant effects. Intertemporal effects describe the influence of changes in wage and price which occur at different dates. They have to be integrated over the planning period. A striking feature of continuous time models is that contemporaneous income effect are negligible, and therefore do not appear in the above formulas. The reason is that income effects have to be summed over a finite period in order to become significant. Contemporaneous substitution effects, to the contrary, are of the same order of magnitude than sums of intertemporal effects. When considered in isolation, they can be interpreted, as we saw, as the limit-effect of a purely instantaneous shock. One of the advantages of continuous time models is therefore that it allows a rigorous analysis of a purely instantaneous shock, showing that this generates only substitution effects. In a discrete time model an instantaneous shock always mingles substitution and income effects, even if many authors have been tempted to argue (rightly but not quite rigorously) that income effects should be considered as second order.

x_0-constant effects are therefore contemporaneous substitution effects. We may add, refering to the distinction introduced[5] by Frisch(1959), that they are "specific" substitution effects. The "general" substitution effects of variations of w_t and q_t operate through x_0. In a continuous time model they are negligible as are the contemporaneous income effects.

A technical point regarding substitution effects should be noted here. There are two reasons why matrix $\mathcal{N}$ is not the true Slutsky matrix. The first one is that the true prices of consumption and leisure are q_t and $w_t q_t$ rather than q_t and w_t. The second reason is that the entries in matrix $\mathcal{N}$ are elasticities rather than partial derivatives. The true substitution matrix is therefore

$$\begin{pmatrix} \frac{c_t^*}{q_t^*}(n_{cx} - n_{cw}) & \frac{c_t^*}{w_t^* q_t^*} n_{cw} \\ \frac{l_t^*}{q_t^*}(n_{lx} - n_{lw}) & \frac{l_t^*}{w_t^* q_t^*} n_{lw} \end{pmatrix} = \frac{1}{\eta + (1-\sigma)\xi} \begin{pmatrix} -\frac{c_t^*}{q_t^*}(\sigma\eta + (1-\sigma)\xi) & \frac{c_t^*}{w_t^* q_t^*}(1-\sigma)\xi \\ \frac{l_t^*}{q_t^*}(1-\sigma) & -\frac{l_t^*}{w_t^* q_t^*} \end{pmatrix}$$

The own-price effects on the diagonal are negative. As $w_t^* l_t^* / c_t^* = \xi$, cross-effects are indeed equal.

Among the set of elasticites which describe the contemporaneous substitution effects, the most important is the elasticity giving the effect of wages on current labor supply :

$$n_{hw} = -\frac{l}{1-l} n_{lw} = \frac{l}{1-l} \frac{1}{\eta + (1-\sigma)\xi}$$

This x_0-constant elasticity is the one which is usually estimated in applied work. It is often refered to flatly as the intertemporal elasticity of substitution of labor supply. The terminology may here create some confusion, as we would rather insist on its contemporaneous character, and reserve the intertemporal denomination to the effects of changes in wages occuring at different dates. This, of course, is not meant to deny that all these elasticities, taken together, describe a unique process of intertemporal substitution, nor that n_{hw} plays a leading role in this process.

More important that this semantic issue is the use which can be made of this elasticity. It has been noted by many authors, among which MaCurdy(1981) and, recently, McLaughlin(1995) that it can be used in an "evolutionnist", rather than in a comparative dynamics, perspective. It is then used to describe the effects of changes in wages along the life-cycle of an agent. Let us sketch the method.

The analysis of consumer behavior leads to the Frischian labor supply function, say, $h_t = g(w_t, x_t) = g\left(w_t, x_0 e^{(\rho+n)t} q_t\right)$. For constant x_0, that is along a given life-cycle profile, this function tells us how labor supply depends on the current levels of wages, intertemporal prices and on time. To clarify let us consider the special case where the rate of interest is constantly equal to $\rho + n$. Then $e^{(\rho+n)t} q_t = 1$. For given x_0 , we are left with a function $h_t = g(w_t)$. Along the given life-cycle profile, labor supply depends solely on the wage rate. It is high when the wage rate is high and conversely. Elasticity n_{hw} is attached to this last g function. It describes the intensity of this intertemporal substitution.

This evolutionnist interpretation of n_{hw} however is not the one we use here. What is of interest for us, in a comparative dynamics perspective, is how the whole profile of labor-supply shifts when the wage profile shifts. n_{hw} is then an important part of the story.

While keeping the same perspective, the elasticities in matrix $\mathcal{N}$ may be used in a slightly different way, to describe the evolution of growth rates rather than levels. Consider for instance consumption. c_t is a function of w_t and x_t. It follows that

$$\dot{c}_t/c_t = n_{cw}\dot{w}_t/w_t + n_{cx}\dot{x}_t/x_t = n_{cw}\dot{w}_t/w_t + n_{cx}\left(\rho + n - r_t\right)$$

and therefore

$$\delta\left(\dot{c}_t/c_t\right) = n_{cw}\delta\left(\dot{w}_t/w_t\right) - n_{cx}\delta r_t$$

n_{cx} thus describes the influence of a change in interest rate on the rate of growth of consumption. Similarly, n_{cw} describes the influence of a change in the rate of growth of wages.

Lastly, consideration of cross-elasticities shows that current leisure and consumption are substitutes if $\sigma < 1$ and complements in the opposite case. Thus a high value of σ, that is easy intertemporal substitution in consumption, implies that consumption and leisure at the same date become complements.

Let us now turn to intertemporal effects. They affect in the same way consumption and leisure. Through the income effect, an increase in future wages induces more consumption and more leisure. An increase in intertemporal prices affects both wage income and consumption expenses. As s is smaller than α, the second effect dominates, so that the overall income effect of an increase in intertemporal prices is negative. Lastly, an increase in prices or wages induces, through substitution effects[6], more consumption and leisure today. Predictably, the income and substitution effects of intertemporal prices, that is of interest rates, play in opposite directions. Of course, the higher σ, the stronger substitution effects are.

Substitution and income effects of a technological shock

We now quantify all these substitution and income effects. To this end, we calculate the changes in wages and intertemporal prices following a technological shock. We then determine and sum their effects on consumption and leisure. Admittedly, this analysis of the consumer's behavior is not a constructive method for solving the model, since we use in the first place the solution to determine the evolution of prices and wages. It is however a useful method to interpret results.

The evolution of capital intensity is the following :

$$\delta\widetilde{k}_t/\widetilde{k}_t = e^{-\theta t}\delta\widetilde{k}_0/\widetilde{k}_0 + b_k\frac{e^{-\theta t} - e^{-\lambda t}}{\lambda - \theta}\delta A_0/A_0$$

It follows :

$$\frac{\delta r_t}{r_t} = \left(- \left(1 + \frac{b_k}{\theta - \lambda} \right) T_{rk} e^{-\theta t} + \left(\frac{b_k}{\theta - \lambda} T_{rk} + T_{ra} \right) e^{-\lambda t} \right) \frac{\delta A_0}{A_0}$$

$$\frac{\delta w_t}{w_t} = \left(- \left(1 + \frac{b_k}{\theta - \lambda} \right) T_{wk} e^{-\theta t} + \left(\frac{b_k}{\theta - \lambda} T_{wk} + T_{wa} \right) e^{-\lambda t} \right) \frac{\delta A_0}{A_0}$$

The calculation of the variation of the intertemporal price is slightly more complicated as we have to sum the variations of interest rates over the horizon of the agent.

$$\frac{\delta q_t}{q_t} = -\delta R_t = - \int_0^t \delta r_s \ ds = -(\rho' + g) \int_0^t \frac{\delta r_s}{r_s} ds =$$

$$= (\rho' + g) \left(\left(1 + \frac{b_k}{\theta - \lambda} \right) T_{rk} \frac{1 - e^{-\theta t}}{\theta} - \left(\frac{b_k}{\theta - \lambda} T_{rk} + T_{ra} \right) \frac{1 - e^{-\lambda t}}{\lambda} \right) \frac{\delta A_0}{A_0}$$

The technological shock implies in general a temporary increase in the marginal productivity of capital and therefore in the interest rate. The effect on real wages is ambiguous. It is definitely positive in the long run. It is usually positive in the short run, but could be negative if the elasticity of labor supply is very low, so that a large variation of the real wage would be required in order to ensure continuing full employment after a shock. This happens if σ_f is very low. Lastly, we observe a permanent decrease of intertemporal prices, following the increase in the interest rates. Typical evolutions of wages and interest rates have been plotted on figure 4.

We now calculate the following sums, in order to analyze cumulated effects :

$$S_q = \rho' \int_0^\infty e^{-\rho' t} \frac{\delta q_t}{q_t} \ dt =$$

$$= \left(\frac{\rho' + g}{\rho' + \theta} \left(1 + \frac{b_k}{\theta - \lambda} \right) T_{rk} - \frac{\rho' + g}{\rho' + \lambda} \left(\frac{b_k}{\theta - \lambda} T_{rk} + T_{ra} \right) \right) \frac{\delta A_0}{A_0}$$

$$S_w = \rho' \int_0^\infty e^{-\rho' t} \frac{\delta w_t}{w_t} \ dt =$$

$$= \left(- \frac{\rho'}{\rho' + \theta} \left(1 + \frac{b_k}{\theta - \lambda} \right) T_{wk} + \frac{\rho'}{\rho' + \lambda} \left(\frac{b_k}{\theta - \lambda} T_{wk} + T_{wa} \right) \right) \frac{\delta A_0}{A_0}$$

These expressions, in general, have ambiguous signs, so that there is not much hope to obtain definite results concerning the various income and substitution effects. One general result however can be reached concerning the sum of all income effects.

Proposition

After a technological shock, the representative consumer experiences a relative compensated variation in wealth

$$\left(-S_q + \frac{1-\alpha}{\alpha - s} S_w\right) \frac{\delta A_0}{A_0} = \frac{1-\alpha}{\alpha} \frac{\rho' + g + \mu}{\rho' + \lambda} \frac{\delta A_0}{A_0}$$

This simple expression follows directly from our previous analysis of the value-function, which was made in the centralized version of the model. It is however interesting to check that it does correspond, in the pure consumer's problem, to the sum of all income effects. As a matter of fact, the derivation of the formula is simpler in the decentralized version of the model where it follows from the factor-price frontier. The idea is that all national income is eventually redistributed to the consumer, be it through wages or interest rates.

We thus find the reassuring result that the representative consumer always benefits from a positive technological shock, and that the more persistent the shock, the larger the gain.

This is unfortunately the only general result we have been able to obtain. Intertemporal substitution effects cannot be given a simple form, and their sign is ambiguous. Even for standard parameters values, the sign of S_q is indeterminate. This is a consequence of the overshooting phenomenon which we already identified. After a permanent shock, interest rate stay permanently above their benchmark value. S_q then is negative. When the shock is temporary, the interest rate quickly becomes inferior to its long run value. As is pictured on figure 7, S_q become positive as soon as λ is above a rather low critical level. The influence of interest rate is then reversed. The effects of future low interest rates dominate the effects of the short run high rates.

Figure 7

Our analysis however allows us to quantify numerically the various effects. Let us consider again the calibration of King-Plosser-Rebelo(1988).

The precise decomposition is given in the following table :

	$\delta c_0/c_0$	$\delta h_0/h_0$
contemp. subst. effect of $\delta w_0/w_0$	0	0.205
subst. effects of future $\delta w_t/w_t$	0.028	-0.011
subst. effects of future $\delta q_t/q_t$	0.018	-0.007
income effects of future $\delta w_t/w_t$	0.069	-0.028
income effects of future $\delta q_t/q_t$	-0.002	0.001
overall effect	0.112	0.160

The initial increase in wages induces more labor, but has no effect on consumption because of the unit elasticity of substitution. The positive overall income effect of the technological shock induces the agent to consume more and work less. Intertemporal substitution effects associated to changes in interest rates dominate by far contrary income effects. The surprising element is that the total effects of interest rates changes is to induce the agent to save less and to take more leisure.

The bottom line is that consumption increases, mainly as a result of the income effect of future wages, while labor supply increases, mainly because of the increase in current wages.

Notes

1 We could alternatively make the utilitarist assumption that the representative agent maximizes $\int_0^\infty e^{-\rho t} N_t U(c_t, l_t)\ dt$. This would amount to substituting $\rho - n$ to ρ.

2 See McLaughlin(1995) for a recent reference.

3 Other parameters are fixed at the values which take in the calibration which we consider below.

4 See Killingsworth(1983) for a somewhat informal decomposition along these lines.

5 See e.g. Phlips(1974) or Barten-Böhm(1982) for a definition of specific and general substituion effects.

6 Intertemporal substitution effects are general substitution effects. Time separability of preferences implies that specific intertemporal substitution effects are zero.

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Appendix

Concavity of the utility function

The first derivatives of the utility functions are

$U'_c(c,l) = c^{-1/\sigma}v(l)^{1-1/\sigma}$

$U'_l(c,l) = c^{1-1/\sigma}v(l)^{-1/\sigma}v'(l) = c^{1-1/\sigma}v(l)^{1-1/\sigma}(v'(l)/v(l))$

The matrix of second derivatives, calculated at the stationary poiint is

$$\begin{pmatrix} -(1/\sigma)c^{-1/\sigma}v(l)^{1-1/\sigma}/c & (1-1/\sigma)\xi c^{-1/\sigma}v(l)^{1-1/\sigma}/l \\ (1-1/\sigma)c^{1-1/\sigma}v(l)^{1-1/\sigma}(v'(l)/v(l))/c & ((1-1/\sigma)\xi-\eta)c^{1-1/\sigma}v(l)^{1-1/\sigma}(v'(l)/v(l))/l \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{1}{\sigma}\frac{U}{c^2} & \left(1-\frac{1}{\sigma}\right)\xi\frac{U}{cl} \\ \left(1-\frac{1}{\sigma}\right)\xi\frac{U}{cl} & \left(\left(1-\frac{1}{\sigma}\right)\xi-\eta\right)\xi\frac{U}{l^2} \end{pmatrix}$$

The utility function is concave iff $U''_{cc} \leq 0$ and $U''_{cc}U''_{ll} - \left(U''_{cl}\right)^2 \geq 0$. The first condition always holds. The second is satisfied iff $-\frac{1}{\sigma}\left(\left(1-\frac{1}{\sigma}\right)\xi-\eta\right)\xi - \left(1-\frac{1}{\sigma}\right)^2\xi^2 = -\frac{\xi}{\sigma}((\sigma-1)\xi-\eta) \geq 0$ that is iff $\eta+(1-\sigma)\xi \geq 0$

As usual, concavity implies $U''_{ll} \leq 0$, that is $\sigma\eta+(1-\sigma)\xi \geq 0$. This condition ensures concavity of function $v(l)^{1-1/\sigma}$. This condition is necessary, but not sufficient to ensure concavity of function $U(c,l) = (cv(l))^{1-1/\sigma}/(1-1/\sigma)$. We may add that concavity of function $v(l)$, that is $\xi-\eta \leq 0$, is neither necessary nor sufficient.

Linearization

We use the well-known relationships which give the relative variations of the wage and interest rates. In our framework they are :

$\frac{\delta\widetilde{w}}{\widetilde{w}} = \frac{\alpha}{\sigma_f}\frac{\delta\left(\widetilde{k}/(1-l)\right)}{\widetilde{k}/(1-l)} = \frac{\alpha}{\sigma_f}\left(\frac{\delta\widetilde{k}}{\widetilde{k}}+\frac{\delta l}{1-l}\right) = \frac{\alpha}{\sigma_f}\left(\frac{\delta\widetilde{k}}{\widetilde{k}}+\frac{(1-s)\xi}{1-\alpha}\frac{\delta l}{l}\right)$

$\frac{\delta(r+\mu)}{r+\mu} = -\frac{1-\alpha}{\sigma_f}\frac{\delta\left(\widetilde{k}/(1-l)\right)}{\widetilde{k}/(1-l)} = -\frac{1-\alpha}{\sigma_f}\left(\frac{\delta\widetilde{k}}{\widetilde{k}}+\frac{(1-s)\xi}{1-\alpha}\frac{\delta l}{l}\right)$

Linearization of the static relationships

$\widetilde{c}^{-1/\sigma}v(l)^{1-1/\sigma} = \widetilde{x}$

$\widetilde{c}v'(l)/v(l) = \widetilde{w} = \left(f\left(\frac{\widetilde{k}}{1-l}\right) - \frac{\widetilde{k}}{1-l}f'\left(\frac{\widetilde{k}}{1-l}\right)\right)$

yields

$-\frac{1}{\sigma}\frac{\delta\widetilde{c}}{\widetilde{c}} + \frac{\sigma-1}{\sigma}\xi\frac{\delta l}{l} = \frac{\delta\widetilde{x}}{\widetilde{x}}$

$\frac{\delta\widetilde{c}}{\widetilde{c}} - \eta\frac{\delta l}{l} = \frac{\delta\widetilde{w}}{\widetilde{w}} = \frac{\alpha}{\sigma_f}\left(\frac{\delta\widetilde{k}}{\widetilde{k}}+\frac{(1-s)\xi}{1-\alpha}\frac{\delta l}{l}\right)$

or

$$\frac{\delta\widetilde{c}}{\widetilde{c}} - \left(\eta + \frac{\alpha(1-s)\xi}{\sigma_f(1-\alpha)}\right)\frac{\delta l}{l} = \frac{\alpha}{\sigma_f}\frac{\delta\widetilde{k}}{\widetilde{k}}$$

It follows that

$$(\eta + (1-\sigma)\xi)\frac{\delta\widetilde{c}}{\widetilde{c}} = (1-\sigma)\xi\frac{\delta\widetilde{w}}{\widetilde{w}} - \sigma\eta\frac{\delta\widetilde{x}}{\widetilde{x}}$$

$$(\eta + (1-\sigma)\xi)\frac{\delta l}{l} = -\frac{\delta\widetilde{w}}{\widetilde{w}} - \sigma\frac{\delta\widetilde{x}}{\widetilde{x}}$$

i.e.

$$\left(\eta + (1-\sigma)\xi + \frac{\alpha(1-s)\xi}{\sigma_f(1-\alpha)}\right)\frac{\delta\widetilde{c}}{\widetilde{c}} = (1-\sigma)\xi\frac{\alpha}{\sigma_f}\frac{\delta\widetilde{k}}{\widetilde{k}} - \sigma\left(\eta + \frac{\alpha(1-s)\xi}{\sigma_f(1-\alpha)}\right)\frac{\delta\widetilde{x}}{\widetilde{x}}$$

$$\left(\eta + (1-\sigma)\xi + \frac{\alpha(1-s)\xi}{\sigma_f(1-\alpha)}\right)\frac{\delta l}{l} = -\frac{\alpha}{\sigma_f}\frac{\delta\widetilde{k}}{\widetilde{k}} - \sigma\frac{\delta\widetilde{x}}{\widetilde{x}}$$

Let us now linearize the dynamic conditions

$$\dot{\widetilde{x}}/\widetilde{x} = \rho + \mu + n - f'\left(\widetilde{k}/(1-l)\right) + \frac{1}{\sigma}\frac{\dot{A}}{A}$$

$$\dot{\widetilde{k}} = (1-l)f\left(\widetilde{k}/(1-l)\right) - \left(\mu + n + \dot{A}/A\right)\widetilde{k} - \widetilde{c}$$

We obtain

$$\frac{\delta\dot{\widetilde{k}}}{\widetilde{k}} = \frac{1}{k^*}\delta\dot{\widetilde{k}} = \frac{1}{k^*}\left[\left(f'\left(\frac{\widetilde{k}^*}{1-l^*}\right) - (\mu + n + \gamma)\right)\delta\widetilde{k} + \left(f\left(\frac{\widetilde{k}^*}{1-l^*}\right) - \frac{\widetilde{k}^*}{1-l^*}f'\left(\frac{\widetilde{k}^*}{1-l^*}\right)\right)\delta(1-l) - \delta\widetilde{c}\right] =$$

$$= \left(\rho + \frac{1-\sigma}{\sigma}\gamma\right)\frac{\delta\widetilde{k}}{\widetilde{k}} - \frac{\widetilde{w}^* l^*}{\widetilde{k}^*}\frac{\delta l}{l} - \frac{\widetilde{c}^*}{\widetilde{k}^*}\frac{\delta\widetilde{c}}{\widetilde{c}} =$$

$$= \left(\rho + \frac{1-\sigma}{\sigma}\gamma\right)\frac{\delta\widetilde{k}}{\widetilde{k}} - \xi\frac{\widetilde{c}^*}{\widetilde{k}^*}\frac{\delta l}{l} - \frac{\widetilde{c}^*}{\widetilde{y}^*}\frac{\widetilde{y}^*}{\widetilde{k}^*}\frac{\delta\widetilde{c}}{\widetilde{c}} =$$

$$= \left(\rho + \frac{1-\sigma}{\sigma}\gamma\right)\frac{\delta\widetilde{k}}{\widetilde{k}} - \xi(1-s)\frac{r^*+\mu}{\alpha}\frac{\delta l}{l} - (1-s)\frac{r^*+\mu}{\alpha}\frac{\delta\widetilde{c}}{\widetilde{c}}$$

as well as

$$\frac{\delta\dot{\widetilde{x}}}{\widetilde{x}} = -\delta r + \frac{1}{\sigma}\frac{\delta\dot{A}}{A} = \frac{1-\alpha}{\sigma_f}(r^* + \mu)\left(\frac{\delta\widetilde{k}}{\widetilde{k}} + \frac{(1-s)\xi}{1-\alpha}\frac{\delta l}{l}\right) + \frac{1}{\sigma}\frac{\delta\dot{A}}{A}$$

Letting $\rho' = \rho + \frac{1-\sigma}{\sigma}\gamma$, we have

$$\begin{pmatrix}\delta\dot{\widetilde{k}}/\widetilde{k} \\ \delta\dot{\widetilde{x}}/\widetilde{x}\end{pmatrix} = \begin{pmatrix}\rho' & 0 \\ \dfrac{1-\alpha}{\sigma_f}(r^* + \mu) & 0\end{pmatrix}\begin{pmatrix}\delta\widetilde{k}/\widetilde{k} \\ \delta\widetilde{x}/\widetilde{x}\end{pmatrix} +$$

$$+ \begin{pmatrix}-(1-s)\dfrac{r^*+\mu}{\alpha} & -\xi(1-s)\dfrac{r^*+\mu}{\alpha} \\ 0 & \xi(1-s)\dfrac{r^*+\mu}{\sigma_f}\end{pmatrix}\begin{pmatrix}\delta\widetilde{c}/\widetilde{c} \\ \delta l/l\end{pmatrix} + \begin{pmatrix}-1 \\ 1/\sigma\end{pmatrix}\left(\delta\dot{A}/A\right)$$

$$= J\begin{pmatrix}\delta\widetilde{k}/\widetilde{k} \\ \delta\widetilde{x}/\widetilde{x}\end{pmatrix} + \begin{pmatrix}-1 \\ 1/\sigma\end{pmatrix}\left(\delta\dot{A}/A\right)$$

where

$$J=\begin{pmatrix} \rho'-(1-s)\frac{r^*+\mu}{\alpha}(m_{ck}+\xi m_{lk}) & -(1-s)\frac{r^*+\mu}{\alpha}(m_{cx}+\xi m_{lx}) \\ (1-\alpha)\frac{r^*+\mu}{\sigma_f}\left(1+\frac{(1-s)\xi}{1-\alpha}m_{lk}\right) & (1-s)\xi\frac{r^*+\mu}{\sigma_f}m_{lx} \end{pmatrix}$$

or, taking into account the structure of matrix $\mathcal{M}$

$$J=\begin{pmatrix} \rho'-(1-s)\xi\frac{r^*+\mu}{\alpha}\sigma m_{lk} & -(1-s)\frac{r^*+\mu}{\alpha}\left(\eta+\xi+\frac{(1-s)\xi\alpha}{(1-\alpha)\sigma_f}\right)\frac{\sigma\sigma_f}{\alpha}m_{lk} \\ (1-\alpha)\frac{r^*+\mu}{\sigma_f}\left(1+\frac{(1-s)\xi}{1-\alpha}m_{lk}\right) & (1-s)\xi\frac{r^*+\mu}{\alpha}\sigma m_{lk} \end{pmatrix}$$

The saddle-point property

The trace of matrix J is equal to ρ'. Its determinant is

$$\begin{aligned}
&(1-s)\tfrac{r^*+\mu}{\alpha}\sigma m_{lk}\left\{\left(\rho'-(1-s)\xi\tfrac{r^*+\mu}{\alpha}\sigma m_{lk}\right)\xi+\right.\\
&\left.+\left(\eta+\xi+\tfrac{(1-s)\xi\alpha}{(1-\alpha)\sigma_f}\right)(1-\alpha)\tfrac{r^*+\mu}{\alpha}\left(1+\tfrac{(1-s)\xi}{1-\alpha}m_{lk}\right)\right\}=\\
&=(1-s)\tfrac{r^*+\mu}{\alpha}\sigma m_{lk}\left\{\rho'\xi+(1-\alpha)\left(\eta+\xi+\tfrac{(1-s)\xi\alpha}{(1-\alpha)\sigma_f}\right)\tfrac{r^*+\mu}{\alpha}+\right.\\
&\left.+(1-s)\xi\tfrac{r^*+\mu}{\alpha}m_{lk}\left(\sigma\xi-\left(\eta+\xi+\tfrac{(1-s)\xi\alpha}{(1-\alpha)\sigma_f}\right)\right)\right\}=\\
&=(1-s)\tfrac{r^*+\mu}{\alpha}\sigma m_{lk}\left\{\rho'\xi+(1-\alpha)\left(\eta+\xi+\tfrac{(1-s)\xi\alpha}{(1-\alpha)\sigma_f}\right)\tfrac{r^*+\mu}{\alpha}-(1-s)\xi\tfrac{r^*+\mu}{\alpha}\tfrac{\alpha}{\sigma_f}\right\}=\\
&=(1-s)\tfrac{r^*+\mu}{\alpha}\sigma m_{lk}\left\{\rho'\xi+(1-\alpha)(\eta+\xi)\tfrac{r^*+\mu}{\alpha}\right\}
\end{aligned}$$

This determinant is negative as $m_{lk}<0$ and $\eta+\xi>0$, from the concavity assumption.

The saddle-point property is thus satisfied. The system has a negative eigenvalue $r_1=-\theta$ and a positive one $r_2=\rho'-r_1=\rho'+\theta$. Let $(1,v)$ be the eigenvector associated to r_1. We know that $J_{kk}>0$, $J_{kx}>0$, $J_{xx}<0$ whereas J_{xk} has an ambiguous sign (but is usually positive). In any case, $v=-(J_{kk}+\theta)/J_{kx}$ is negative.

Solution of the model

We solve the system

$$\begin{pmatrix}\delta\dot{\tilde{k}}/\tilde{k}\\ \delta\dot{\tilde{x}}/\tilde{x}\end{pmatrix}=J\begin{pmatrix}\delta\tilde{k}/\tilde{k}\\ \delta\tilde{x}/\tilde{x}\end{pmatrix}+\begin{pmatrix}-1\\ 1/\sigma\end{pmatrix}\left(\delta\dot{A}/A\right)$$

where

$$\frac{\delta A_t}{A_t}=e^{-\lambda t}\frac{\delta A_0}{A_0}$$

The solution is the sum of a particular solution and of the general solution of the homogenous system. The characteristic polynomial of matrix J is $\varphi(r)=(r+\theta)(r-r_2)$.

We retain a particular solution of the following form

$$\begin{pmatrix} \delta\widetilde{k}_t/\widetilde{k}_t \\ \delta\widetilde{x}_t/\widetilde{x}_t \end{pmatrix} = m\frac{\delta A_t}{A_t} = \begin{pmatrix} m_k \\ m_x \end{pmatrix} \frac{\delta A_t}{A_t}$$

Vector m satisfies

$$(J + \lambda I)m = \lambda \begin{pmatrix} -1 \\ 1/\sigma \end{pmatrix}$$

and we have

$$m = \lambda(J + \lambda I)^{-1} \begin{pmatrix} -1 \\ 1/\sigma \end{pmatrix} = \frac{\lambda}{\varphi(-\lambda)} \begin{pmatrix} J_{xx} + \lambda & -J_{kx} \\ -J_{xk} & J_{kk} + \lambda \end{pmatrix} \begin{pmatrix} -1 \\ 1/\sigma \end{pmatrix}$$

or

$$\begin{pmatrix} m_k \\ m_x \end{pmatrix} = \frac{\lambda}{(\lambda-\theta)(\lambda+r_2)} \begin{pmatrix} -J_{xx} - \lambda - J_{kx}/\sigma \\ J_{xk} + (J_{kk} + \lambda)/\sigma \end{pmatrix}$$

The general solution is

$$\begin{pmatrix} \delta\widetilde{k}_t/\widetilde{k}_t \\ \delta\widetilde{x}_t/\widetilde{x}_t \end{pmatrix} = h_1 e^{-\theta t} V_1 + h_2 e^{r_2 t} V_2 + e^{-\lambda t} m \frac{\delta A_0}{A_0}$$

with constants h_1 and h_2 depending on initial conditions

Selecting the non-explosive solution requires $h_2 = 0$. It is convenient to normalize to one the first component of vector V_1. Let v be its second component. The solution becomes

$$\begin{pmatrix} \delta\widetilde{k}_t/\widetilde{k}_t \\ \delta\widetilde{x}_t/\widetilde{x}_t \end{pmatrix} = h_1 e^{-\theta t} \begin{pmatrix} 1 \\ v \end{pmatrix} + e^{-\lambda t} \begin{pmatrix} m_k \\ m_x \end{pmatrix} \frac{\delta A_0}{A_0}$$

The initial condition on the pre-determined variable $\delta\widetilde{k}/\widetilde{k}$ determines constant h_1 and we obtain

$$\delta\widetilde{k}/\widetilde{k} = e^{-\theta t}\delta\widetilde{k}_0/\widetilde{k}_0 + m_k \left(e^{-\lambda t} - e^{-\theta t}\right) \delta A_0/A_0$$

$$== e^{-\theta t}\delta\widetilde{k}_0/\widetilde{k}_0 + b_k \frac{e^{-\theta t} - e^{-\lambda t}}{\lambda - \theta} \delta A_0/A_0$$

where

$$b_k = -(\lambda - \theta)m_k = \frac{\lambda}{\lambda + r_2}\left(J_{xx} + \lambda + J_{kx}/\sigma\right)$$

and

$$\delta\widetilde{x}/\widetilde{x} = e^{-\theta t} v \left(\delta\widetilde{k}_0/\widetilde{k}_0 - m_k \delta A_0/A_0\right) + e^{-\lambda t} m_x \delta A_0/A_0 =$$

$$= v\delta\widetilde{k}_t/\widetilde{k}_t + (m_x - v m_k)\, e^{-\lambda t}\delta A_0/A_0$$

We now look for a more convenient expression.

$$\begin{pmatrix} m_k \\ m_x \end{pmatrix} = \frac{\lambda}{\varphi(-\lambda)} \left[\begin{pmatrix} J_{xx}+\theta & -J_{kx} \\ -J_{xk} & J_{kk}+\theta \end{pmatrix} + (\lambda-\theta) I \right] \begin{pmatrix} -1 \\ 1/\sigma \end{pmatrix} =$$
$$= \frac{\lambda}{\varphi(-\lambda)} \left[\begin{pmatrix} J_{xx}+\theta & -J_{kx} \\ v\,(J_{xx}+\theta) & -vJ_{kx} \end{pmatrix} + (\lambda-\theta) I \right] \begin{pmatrix} -1 \\ 1/\sigma \end{pmatrix}$$

from the definition of eigenvector $(1,v)$ associated to $-\theta$. We thus have $m_x - vm_k =$ $\frac{\lambda}{\varphi(-\lambda)}(\lambda-\theta)(v+1/\sigma) = \frac{\lambda}{\lambda+r_2}(v+1/\sigma) \overset{déf}{=} b_x$

Now the definition of m_k and m_x implies

$$(J_{kk}+\lambda)\, m_k + J_{kx} m_x = -\lambda$$

whereas the definition of the eigenvector implies

$$(J_{kk}+\theta) + J_{kx} v = 0$$

It follows that

$$(\lambda-\theta) m_k + J_{kx}\,(m_x - vm_k) = -\lambda$$

and therefore

$$-b_k + J_{kx} b_x = -\lambda$$

This completes the solution of the system.

The values of other variables can be expressed as functions of the state variables $\delta\tilde{k}_t/\tilde{k}_t$ and $\delta A_t/A_t$. We obtain :

$$\frac{\delta C}{C} = \frac{\delta\tilde{c}}{\tilde{c}} + \frac{\delta A}{A} = m_{ck}\frac{\delta\tilde{k}}{\tilde{k}} + m_{cx}\frac{\delta\tilde{x}}{\tilde{x}} + \frac{\delta A}{A} = (m_{ck} + m_{cx} v)\frac{\delta\tilde{k}}{\tilde{k}} + m_{cx} b_x \frac{\delta A}{A} + \frac{\delta A}{A}$$

$$\frac{\delta l}{l} = m_{lk}\frac{\delta\tilde{k}}{\tilde{k}} + m_{lx}\frac{\delta\tilde{x}}{\tilde{x}} = (m_{lk} + m_{lx} v)\frac{\delta\tilde{k}}{\tilde{k}} + m_{lx} b_x \frac{\delta A}{A}$$

$$\frac{\delta Y}{Y} = \frac{\delta\tilde{y}}{\tilde{y}} + \frac{\delta A}{A} = \alpha\frac{\delta\tilde{k}}{\tilde{k}} - (1-\alpha)\frac{\delta l}{1-l} + \frac{\delta A}{A} = \alpha\frac{\delta\tilde{k}}{\tilde{k}} - (1-s)\xi\frac{\delta l}{l} + \frac{\delta A}{A} =$$

$$= (\alpha - (1-s)\xi\,(m_{lk} + m_{lx} v))\frac{\delta\tilde{k}}{\tilde{k}} - (1-s)\xi m_{lx} b_x \frac{\delta A}{A} + \frac{\delta A}{A}$$

$$\frac{\delta(r+\mu)}{r+\mu} = \frac{r^*}{r^*+\mu}\frac{\delta r}{r} = -\frac{1-\alpha}{\sigma_f}\frac{\delta\left(\tilde{k}/(1-l)\right)}{\tilde{k}/(1-l)} = -\frac{1-\alpha}{\sigma_f}\left(\frac{\delta\tilde{k}}{\tilde{k}} + \frac{(1-s)\xi}{1-\alpha}\frac{\delta l}{l}\right)$$

and therefore

$$\frac{\delta r}{r} = -\frac{r^*+\mu}{r^*}\frac{1-\alpha}{\sigma_f}\left(\left(1 + \frac{(1-s)\xi}{1-\alpha}(m_{lk} + m_{lx} v)\right)\frac{\delta\tilde{k}}{\tilde{k}} + \frac{(1-s)\xi}{1-\alpha} m_{lx} b_x \frac{\delta A}{A}\right)$$

$$\frac{\delta w}{w} = \frac{\delta\tilde{w}}{\tilde{w}} + \frac{\delta A}{A} =$$
$$= \frac{\alpha}{\sigma_f}\left(\left(1 + \frac{(1-s)\xi}{1-\alpha}(m_{lk} + m_{lx} v)\right)\frac{\delta\tilde{k}}{\tilde{k}} + \frac{(1-s)\xi}{1-\alpha} m_{lx} b_x \frac{\delta A}{A}\right) + \frac{\delta A}{A}$$

Lastly, we obtain the relative variation of investment. From $I = \left(\dot{K}/K + \mu\right) K =$

$\left(\dot{\tilde{k}}/\tilde{k}+\dot{A}/A+n+\mu\right)\left(\tilde{k}AN\right)$ we deduce

$$\frac{\delta I}{I}=\frac{\delta\dot{\tilde{k}}/\tilde{k}+\delta\dot{A}/A}{\mu+n+\gamma}+\frac{\delta\tilde{k}}{\tilde{k}}+\frac{\delta A}{A}$$

Now

$$\frac{\delta\dot{\tilde{k}}}{\tilde{k}}=-\theta e^{-\theta t}\left(\frac{\delta\tilde{k}_0}{\tilde{k}_0}-m_k\frac{\delta A_0}{A_0}\right)-\lambda e^{-\lambda t}m_k\frac{\delta A_0}{A_0}=$$

$$=-\theta\frac{\delta\tilde{k}}{\tilde{k}}+(\theta-\lambda)m_k\frac{\delta A}{A}$$

We thus obtain

$$\frac{\delta I}{I}=\frac{-\theta\delta\tilde{k}/\tilde{k}+b_k\delta A/A-\lambda\delta A/A}{g+\mu}+\frac{\delta\tilde{k}}{\tilde{k}}+\frac{\delta A}{A}=$$

$$=\frac{g+\mu-\theta}{g+\mu}\frac{\delta\tilde{k}}{\tilde{k}}+\left(1+\frac{b_k-\lambda}{g+\mu}\right)\frac{\delta A}{A}$$

The sign of initial effects in the case of a permanent shock

Let $\psi=(1-s)\xi/(1-\alpha)$ denote the leisure-labor ratio and $X=\eta+\xi+\alpha\psi/\sigma_f$ the term which appears in the denominator of matrix $\mathcal{M}$. Then

$$J_{kk}=\rho'+(1-s)\xi\sigma\frac{r^*+\mu}{\sigma_f}\frac{1}{X}$$

$$J_{kx}=(1-s)\frac{r^*+\mu}{\alpha}\left(\eta+\xi+\psi\frac{\alpha}{\sigma_f}\right)\frac{\sigma}{X}$$

$$J_{xk}=(1-\alpha)\frac{r^*+\mu}{\sigma_f}\left(1-\psi\frac{\alpha}{\sigma_f}\frac{1}{X}\right)$$

$$J_{xx}=-(1-s)\xi\sigma\frac{r^*+\mu}{\sigma_f}\frac{1}{X}$$

In the general case, initial effects are

$$\frac{\delta x_0}{x_0}=\left(b_x-v-\frac{1}{\sigma}\right)\frac{\delta A_0}{A_0}=\left(\frac{\lambda}{\lambda+r_2}\left(v+\frac{1}{\sigma}\right)-v-\frac{1}{\sigma}\right)\frac{\delta A_0}{A_0}=$$

$$=-\frac{r_2}{\lambda+r_2}\left(v+\frac{1}{\sigma}\right)\frac{\delta A_0}{A_0}$$

$$\frac{\delta C_0}{C_0}=\left(1+m_{cx}b_x-m_{ck}-m_{cx}v\right)\frac{\delta A_0}{A_0}=-m_{cx}\left(v-b_x-\frac{1-m_{ck}}{m_{cx}}\right)\frac{\delta A_0}{A_0}=$$

$$=-m_{cx}\left(v-b_x+\frac{X-(1-\sigma)\xi\frac{\alpha}{\sigma_f}}{\sigma\left(\eta+\psi\frac{\alpha}{\sigma_f}\right)}\right)\frac{\delta A_0}{A_0}=-m_{cx}\left(v-\frac{\lambda}{\lambda+r_2}\left(v+\frac{1}{\sigma}\right)+\frac{1}{\sigma}+\frac{1-\sigma}{\sigma}\frac{\xi\left(1-\frac{\alpha}{\sigma_f}\right)}{\eta+\psi\frac{\alpha}{\sigma_f}}\right)\frac{\delta A_0}{A_0}$$

$$=-m_{cx}\left(\frac{r_2}{\lambda+r_2}\left(v+\frac{1}{\sigma}\right)+\frac{1-\sigma}{\sigma}\frac{\xi\left(1-\frac{\alpha}{\sigma_f}\right)}{\eta+\psi\frac{\alpha}{\sigma_f}}\right)$$

$$\frac{\delta l_0}{l_0}=\left(m_{lx}b_x-m_{lk}-m_{lx}v\right)\frac{\delta A_0}{A_0}=-m_{lx}\left(\frac{r_2}{\lambda+r_2}\left(v+\frac{1}{\sigma}\right)+\frac{\alpha}{\sigma\sigma_f}-\frac{1}{\sigma}\right)\frac{\delta A_0}{A_0}$$

We study their signs in the case of a permanent shock where they are

$$\frac{\delta x_0}{x_0}=-\left(v+\frac{1}{\sigma}\right)\frac{\delta A_0}{A_0}$$

$\frac{\delta C_0}{C_0} = -m_{cx}\left(v + \frac{1}{\sigma} + \frac{1-\sigma}{\sigma}\frac{\xi\left(1-\frac{\alpha}{\sigma_f}\right)}{\eta+\psi\frac{\alpha}{\sigma_f}}\right)$

$\frac{\delta l_0}{l_0} = -m_{lx}\left(v + \frac{\alpha}{\sigma\sigma_f}\right)\frac{\delta A_0}{A_0}$

$\delta x_0/x_0$ is positive iff $v > -1/\sigma$. By definition, $v = (r_1 - J_{kk})/J_{kx}$ and $r_1 = J_{kk} + vJ_{kx}$.We now that r_1 is the smaller root of the characteristic polynomial $\varphi(r) = 0$. As $J_{kx} > 0$, v is the smaller root of polynomial

$$\bar{\varphi}(V) = \varphi(J_{kk} + VJ_{kx}) = \begin{vmatrix} J_{kk} - J_{kk} - VJ_{kx} & J_{kx} \\ J_{xk} & J_{xx} - J_{kk} - VJ_{kx} \end{vmatrix} =$$

$$= J_{kx}\begin{vmatrix} -V & 1 \\ J_{xk} & J_{xx} - J_{kk} - VJ_{kx} \end{vmatrix} = J_{kx}Q(V)$$

if we let $Q(V) = -V(J_{xx} - J_{kk} - VJ_{kx}) - J_{xk} = V^2J_{kx} - V(J_{xx} - J_{kk}) - J_{xk}$

As $J_{kx} > 0$ and $Q(J_{xx}/J_{kx}) = (J_{xx}J_{kk} - J_{kx}J_{xk})/J_{kx} < 0$, v is larger than$-1/\sigma$ iff $Q(-1/\sigma) > 0$ and $-1/\sigma < J_{xx}/J_{kx}$. We therefore have the following equivalence

$$v > -\frac{1}{\sigma} \qquad \Longleftrightarrow \qquad Q\left(-\frac{1}{\sigma}\right) > 0 \quad \text{and} \quad J_{xx} + \frac{1}{\sigma}J_{kx} > 0$$

Similar equivalence apply to $\delta C_0/C_0$ and $\delta l_0/l_0$.

We now calculate

$$J_{xx} + \frac{1}{\sigma}J_{kx} = (1-s)\frac{r^*+\mu}{\alpha}\frac{1}{X}\left(-\xi\sigma\frac{\alpha}{\sigma_f} + \left(\eta + \xi + \psi\frac{\alpha}{\sigma_f}\right)\right) =$$

$$= (1-s)\frac{r^*+\mu}{\alpha}\frac{1}{X}\left(X + \xi\sigma\left(1 - \frac{\alpha}{\sigma_f}\right)\right) = (1-s)\frac{r^*+\mu}{\alpha}\left(1 + \frac{\xi\sigma\left(1-\frac{\alpha}{\sigma_f}\right)}{X}\right) =$$

$$= (1-s)\frac{r^*+\mu}{\alpha}\frac{\eta+\psi\frac{\alpha}{\sigma_f}+\xi\left(1-\frac{\sigma\alpha}{\sigma_f}\right)}{X}$$

$$J_{xx} + \frac{\alpha}{\sigma\sigma_f}J_{kx} = (1-s)\frac{r^*+\mu}{\sigma_f}\frac{1}{X}\left(-\xi\sigma + \left(\eta + \xi + \psi\frac{\alpha}{\sigma_f}\right)\right) = (1-s)\frac{r^*+\mu}{\sigma_f}$$

$$J_{xx} + \left(\frac{1}{\sigma} + \frac{1-\sigma}{\sigma}\frac{\xi\left(1-\frac{\alpha}{\sigma_f}\right)}{\eta+\psi\frac{\alpha}{\sigma_f}}\right)J_{kx} =$$

$$(1-s)\frac{r^*+\mu}{\alpha}\left(1 + \frac{\xi\sigma\left(1-\frac{\alpha}{\sigma_f}\right)}{X} + \frac{1-\sigma}{\sigma}\frac{\xi\left(1-\frac{\alpha}{\sigma_f}\right)}{\eta+\psi\frac{\alpha}{\sigma_f}}\frac{\sigma}{X}\left(\eta + \xi + \psi\frac{\alpha}{\sigma_f}\right)\right) =$$

$$= (1-s)\frac{r^*+\mu}{\alpha}\left(1 + \frac{\xi\left(1-\frac{\alpha}{\sigma_f}\right)}{X}\left(\sigma + (1-\sigma) + \frac{(1-\sigma)\xi}{\eta+\psi\frac{\alpha}{\sigma_f}}\right)\right) =$$

$$= (1-s)\frac{r^*+\mu}{\alpha}\left(1 + \frac{\xi\left(1-\frac{\alpha}{\sigma_f}\right)}{\eta+\psi\frac{\alpha}{\sigma_f}}\right) = (1-s)\frac{r^*+\mu}{\alpha}\frac{\eta+\psi\frac{\alpha}{\sigma_f}+\xi\left(1-\frac{\alpha}{\sigma_f}\right)}{\eta+\psi\frac{\alpha}{\sigma_f}}$$

$$Q\left(-\frac{1}{\sigma}\right)=\frac{1}{\sigma}\left(J_{xx}-J_{kk}+\frac{1}{\sigma}J_{kx}\right)-J_{xk}=$$

$$=\frac{1}{\sigma}\left(-\rho'+(1-s)\frac{r^{*}+\mu}{\alpha}\left(-\frac{\xi\sigma\frac{\alpha}{\sigma_f}}{X}+1+\frac{\xi\sigma\left(1-\frac{\alpha}{\sigma_f}\right)}{X}\right)\right)-(1-\alpha)\frac{r^{*}+\mu}{\sigma_f}+(1-s)\xi\frac{r^{*}+\mu}{\sigma_f}\frac{\alpha}{\sigma_f}\frac{1}{X}=$$

$$=\frac{1}{\sigma}\left(-\rho'+(1-s)\frac{r^{*}+\mu}{\alpha}\right)-(1-\alpha)\frac{r^{*}+\mu}{\sigma_f}+(1-s)\xi\frac{r^{*}+\mu}{\alpha}\frac{1}{X}\left(1-2\frac{\alpha}{\sigma_f}+\left(\frac{\alpha}{\sigma_f}\right)^2\right)==\frac{1}{\sigma}\left(-\rho'+\left(1-\frac{\alpha(}{\rho'+}\right.\right.$$

$$(1-\alpha)\frac{\rho'+g+\mu}{\sigma_f}+(1-s)\xi\frac{\rho'+g+\mu}{\alpha}\frac{1}{X}\left(1-\frac{\alpha}{\sigma_f}\right)^2=$$

$$=\frac{1}{\sigma}\left((1-\alpha)\frac{\rho'+g+\mu}{\alpha}\right)-(1-\alpha)\frac{\rho'+g+\mu}{\sigma_f}+(1-s)\xi\frac{\rho'+g+\mu}{\alpha}\frac{1}{X}\left(1-\frac{\alpha}{\sigma_f}\right)^2=$$

$$=\frac{\rho'+g+\mu}{\alpha\sigma\sigma_f^2X}\left(\sigma_f(1-\alpha)\left(\sigma_f-\alpha\sigma\right)+(1-s)\xi\sigma\left(\sigma_f-\alpha\right)^2\right)$$

$$Q\left(-\frac{\alpha}{\sigma\sigma_f}\right)=\frac{\alpha}{\sigma\sigma_f}\left(J_{xx}-J_{kk}+\frac{\alpha}{\sigma\sigma_f}J_{kx}\right)-J_{xk}=$$

$$=\frac{\alpha}{\sigma\sigma_f}\left(-\rho'-(1-s)\xi\sigma\frac{r^{*}+\mu}{\sigma_f}\frac{1}{X}+(1-s)\frac{r^{*}+\mu}{\sigma_f}\right)-(1-\alpha)\frac{r^{*}+\mu}{\sigma_f}+(1-s)\xi\frac{r^{*}+\mu}{\sigma_f}\frac{\alpha}{\sigma_f}\frac{1}{X}=$$

$$=\frac{\alpha}{\sigma\sigma_f}\left(-\rho'+(1-s)\frac{r^{*}+\mu}{\sigma_f}\right)-(1-\alpha)\frac{r^{*}+\mu}{\sigma_f}=$$

$$=\frac{\alpha}{\sigma\sigma_f}\left(-(\alpha-s)\frac{r^{*}+\mu}{\alpha}+(1-s)\frac{r^{*}+\mu}{\sigma_f}\right)-(1-\alpha)\frac{r^{*}+\mu}{\sigma_f}=$$

$$=\frac{r^{*}+\mu}{\sigma\sigma_f}\left(-(\alpha-s)+(1-s)\frac{\alpha}{\sigma_f}-\sigma(1-\alpha)\right)=\frac{r^{*}+\mu}{\sigma\sigma_f}\left((1-s)\left(\frac{\alpha}{\sigma_f}-1\right)+(1-\alpha)(1-\sigma)\right)$$

and

$$Q\left(\frac{X-(1-\sigma)\xi\frac{\alpha}{\sigma_f}}{\sigma\left(\eta+\psi\frac{\alpha}{\sigma_f}\right)}\right)=\frac{X-(1-\sigma)\xi\frac{\alpha}{\sigma_f}}{\sigma\left(\eta+\psi\frac{\alpha}{\sigma_f}\right)}\left(-\rho'+(1-s)\frac{r^{*}+\mu}{\alpha}-(1-s)\xi\sigma\frac{r^{*}+\mu}{\sigma_f}\frac{1}{X}+(1-s)\frac{r^{*}+\mu}{\alpha}\frac{\xi\left(1-\frac{\alpha}{\sigma_f}\right)}{\eta+\psi\frac{\alpha}{\sigma_f}}\right)$$

$$-(1-\alpha)\frac{r^{*}+\mu}{\sigma_f}+(1-s)\xi\frac{r^{*}+\mu}{\sigma_f}\frac{\alpha}{\sigma_f}\frac{1}{X}=$$

$$=\frac{X-(1-\sigma)\xi\frac{\alpha}{\sigma_f}}{\sigma\left(\eta+\psi\frac{\alpha}{\sigma_f}\right)}\left(-(\alpha-s)\frac{r^{*}+\mu}{\alpha}+(1-s)\frac{r^{*}+\mu}{\alpha}+(1-s)\frac{r^{*}+\mu}{\alpha}\frac{\xi\left(1-\frac{\alpha}{\sigma_f}\right)}{\eta+\psi\frac{\alpha}{\sigma_f}}\right)$$

$$-(1-\alpha)\frac{r^{*}+\mu}{\sigma_f}+(1-s)\xi\frac{r^{*}+\mu}{\sigma_f}\frac{1}{X}\left(-\frac{X-(1-\sigma)\xi\frac{\alpha}{\sigma_f}}{\eta+\psi\frac{\alpha}{\sigma_f}}+\frac{\alpha}{\sigma_f}\right)$$

$$=\frac{X-(1-\sigma)\xi\frac{\alpha}{\sigma_f}}{\sigma\left(\eta+\psi\frac{\alpha}{\sigma_f}\right)}\left((1-\alpha)\frac{r^{*}+\mu}{\alpha}+(1-\alpha)\frac{r^{*}+\mu}{\alpha}\frac{\psi\left(1-\frac{\alpha}{\sigma_f}\right)}{\eta+\psi\frac{\alpha}{\sigma_f}}\right)$$

$$-(1-\alpha)\frac{r^{*}+\mu}{\sigma_f}+(1-s)\xi\frac{r^{*}+\mu}{\sigma_f}\frac{\frac{\alpha}{\sigma_f}-1}{\eta+\psi\frac{\alpha}{\sigma_f}}=$$

$$=\frac{X-(1-\sigma)\xi\frac{\alpha}{\sigma_f}}{\sigma\left(\eta+\psi\frac{\alpha}{\sigma_f}\right)}(1-\alpha)\frac{r^{*}+\mu}{\alpha}\frac{\eta+\psi}{\eta+\psi\frac{\alpha}{\sigma_f}}-(1-\alpha)\frac{r^{*}+\mu}{\sigma_f}\left(1+\frac{\psi\left(\frac{\alpha}{\sigma_f}-1\right)}{\eta+\psi\frac{\alpha}{\sigma_f}}\right)=$$

$$=(1-\alpha)\frac{r^{*}+\mu}{\alpha}\frac{\eta+\psi}{\eta+\psi\frac{\alpha}{\sigma_f}}\left(\frac{X-(1-\sigma)\xi\frac{\alpha}{\sigma_f}}{\sigma\left(\eta+\psi\frac{\alpha}{\sigma_f}\right)}-\frac{\alpha}{\sigma_f}\right)=$$

$= (1-\alpha)\frac{r^*+\mu}{\alpha}\frac{\eta+\psi}{\eta+\psi\frac{\alpha}{\sigma_f}}\left(\left(\eta+\psi\frac{\alpha}{\sigma_f}\right)\left(1-\frac{\sigma\alpha}{\sigma_f}\right)+(1-\sigma)\xi\left(1-\frac{\alpha}{\sigma_f}\right)\right)$

The equation of curve CC is

$\left(\eta+\psi\frac{\alpha}{\sigma_f}\right)\left(1-\frac{\sigma\alpha}{\sigma_f}\right)+(1-\sigma)\xi\left(1-\frac{\alpha}{\sigma_f}\right)=0$

Its slope at the point $\left(\sigma=1,\sigma_f=\alpha\right)$, is $d\sigma_f/d\sigma = \alpha$. Thus curve CC is tangent to the straight line $\sigma_f = \alpha\sigma$.

On the other hand $\sigma_f > \alpha\sigma$ implies $Q(-1/\sigma) > 0$, that is $\delta x_0/x_0 < 0$. Region $\delta x_0/x_0$ includes the half-plane $\sigma_f > \alpha\sigma$, which also means tangency.

The value-function

It is possible to calculate the effects of variations in K_0 and A_0 on the utility level reached by the agent. We obtain

$\delta V_0 = \int_0^{+\infty} e^{-\rho t}\left(U'^*_c\delta c_t + U'^*_l\delta l_t\right)\, dt = \int_0^{+\infty} e^{-\rho t}x_t^* c_t^*\left(\frac{\delta c_t}{c_t}+\xi\frac{\delta l_t}{l_t}\right)\, dt =$

$= x_0^* c_0^*\int_0^{+\infty} e^{-\rho' t}\left(\frac{\delta c_t}{c_t}+\xi\frac{\delta l_t}{l_t}\right)\, dt$

As

$V_0^* = \int_0^{\infty} e^{-\rho t}U\left(c_t^*, l_t^*\right)\, dt = \int_0^{\infty} e^{-\rho t}e^{\frac{\sigma-1}{\sigma}\gamma t}U\left(c_0^*, l^*\right)\, dt = \frac{U(c_0^*,l^*)}{\rho'}$

and

$x_0^* = U_c'\left(c_0^*, l^*\right) = \frac{\sigma-1}{\sigma}\frac{U(c_0^*,l^*)}{c_0^*}$

we obtain

$\delta V_0 = \frac{\sigma-1}{\sigma}\rho' V_0^*\int_0^{+\infty} e^{-\rho' t}\left(\frac{\delta c_t}{c_t}+\xi\frac{\delta l_t}{l_t}\right)\, dt$

On the other hand

$\frac{\delta c_t}{c_t}+\xi\frac{\delta l_t}{l_t} = (T_{ck}+\xi T_{lk})\frac{\delta\tilde{k}_t}{\tilde{k}_t}+(T_{ca}+\xi T_{la})\frac{\delta A_t}{A_t} =$

$= (T_{ck}+\xi T_{lk})\left(e^{-\theta t}\left(\frac{\delta K_0}{K_0}-\frac{\delta A_0}{A_0}\right)+b_k\frac{e^{-\theta t}-e^{-\lambda t}}{\lambda-\theta}\frac{\delta A_0}{A_0}\right)+(T_{ca}+\xi T_{la})\, e^{-\lambda t}\frac{\delta A_0}{A_0}$

and we deduce, by integration

$\delta V_0 = \frac{\sigma-1}{\sigma}\rho' V_0^*\left(Z_K\frac{\delta K_0}{K_0}+Z_A\frac{\delta A_0}{A_0}\right)$

where

$Z_K = \frac{T_{ck}+\xi T_{lk}}{\rho'+\theta}$

$Z_A = (T_{ck}+\xi T_{lk})\left(-\frac{1}{\rho'+\theta}+\frac{b_k}{\lambda-\theta}\left(\frac{1}{\rho'+\theta}-\frac{1}{\rho'+\lambda}\right)\right)+(T_{ca}+\xi T_{la})\frac{1}{\rho'+\lambda}$

$= (T_{ck}+\xi T_{lk})\left(-\frac{1}{\rho'+\theta}+\frac{b_k}{(\rho'+\theta)(\rho'+\lambda)}\right)+(T_{ca}+\xi T_{la})\frac{1}{\rho'+\lambda}$

These two coefficients may be given a much simplerr expression. We have

$Z_k = \frac{m_{ck}+\xi m_{lk}+(m_{cx}+\xi m_{lx})v}{\rho'+\theta}$

The definition of the eigenvector associated with $-\theta$ implies

$$(\rho'+\theta)-(1-s)\frac{\rho'+g+\mu}{\alpha}(m_{ck}+\xi m_{lk})-v(1-s)\frac{\rho'+g+\mu}{\alpha}(m_{cx}+\xi m_{lx})=0$$

and we obtainn

$$Z_k=\frac{\alpha}{(1-s)(\rho'+g+\mu)}$$

Furthermore

$$Z_A=(T_{ck}+\xi T_{lk})\left(-\frac{1}{\rho'+\theta}+\frac{J_{kx}b_x+\lambda}{(\rho'+\theta)(\rho'+\lambda)}\right)+\frac{1+(m_{cx}+\xi m_{lx})b_x}{\rho'+\lambda}=$$

$$=-\frac{\rho'(T_{ck}+\xi T_{lk})}{(\rho'+\theta)(\rho'+\lambda)}+\frac{1}{\rho'+\lambda}+\frac{b_x}{\rho'+\lambda}\left(J_{kx}\frac{T_{ck}+\xi T_{lk}}{\rho'+\theta}+m_{cx}+\xi m_{lx}\right)=$$

$$=\frac{1}{\rho'+\lambda}\left(1-\frac{\rho'\alpha}{(1-s)(\rho'+g+\mu)}\right)+\frac{b_x}{\rho'+\lambda}\left(-(1-s)\frac{\rho'+g+\mu}{\alpha}(m_{cx}+\xi m_{lx})\frac{\alpha}{(1-s)(\rho'+g+\mu)}+m_{cx}+\xi m_{lx}\right)$$

$$=\frac{1}{\rho'+\lambda}\left(1-\frac{\rho'\alpha}{\rho'+(1-\alpha)(g+\mu)}\right)=\frac{1}{\rho'+\lambda}\frac{1-\alpha}{1-s}$$

Substitution and income effects in the consumer's behavior

In differential form, the intertemporal budget constraint is

$$(1-s)\int_0^\infty e^{-\rho't}\left(\frac{\delta q_t}{q_t}+\frac{\delta c_t}{c_t}\right)dt=\frac{\alpha-s}{\rho'}\frac{\delta K_0}{K_0}+(1-\alpha)\int_0^\infty e^{-\rho't}\left(\frac{\delta q_t}{q_t}+\frac{\delta w_t}{w_t}-\frac{(1-s)\xi}{1-\alpha}\frac{\delta l_t}{l_t}\right)dt$$

As $\delta c_t/c_t=n_{cw}\delta w_t/w_t+n_{cx}(\delta x_0/x_0+\delta q_t/q_t)$ and $\delta l_t/l_t=n_{lw}\delta w_t/w_t+n_{lx}(\delta x_0/x_0+\delta q_t/q_t)$, we obtain

$$\int_0^\infty e^{-\rho't}\left((1-s)n_{cx}+(1-s)\xi n_{lx}\right)\,dt\frac{\delta x_0}{x_0}=$$

$$\frac{\alpha-s}{\rho'}\frac{\delta K_0}{K_0}+\int_0^\infty e^{-\rho't}\left(-(\alpha-s)\frac{\delta q_t}{q_t}+(1-\alpha)\frac{\delta w_t}{w_t}\right)\,dt+$$

$$-\int_0^\infty e^{-\rho't}\left[\left((1-s)n_{cx}+(1-s)\xi n_{lx}\right)\frac{\delta q_t}{q_t}+\left((1-s)n_{cw}+(1-s)\xi n_{lw}\right)\frac{\delta w_t}{w_t}\right]dt$$

or

$$-\frac{1}{\rho'}\frac{(1-s)(\eta+\xi)\sigma}{\eta+(1-\sigma)\xi}\frac{\delta x_0}{x_0}=\frac{\alpha-s}{\rho'}\frac{\delta K_0}{K_0}+\int_0^\infty e^{-r't}\left(-(\alpha-s)\frac{\delta q_t}{q_t}+(1-\alpha)\frac{\delta w_t}{w_t}\right)\,dt+$$

$$+\int_0^\infty e^{-\rho't}\left[\frac{(1-s)(\eta+\xi)\sigma}{\eta+(1-\sigma)\xi}\frac{\delta q_t}{q_t}+\frac{(1-s)\sigma\xi}{\eta+(1-\sigma)\xi}\frac{\delta w_t}{w_t}\right]dt$$

The intertemporal budget constraint allows us to determine $\delta x_0/x_0$. We then are able to decompose the initial variations in consumption and leisure in sums of substitution and income effect. Results appear in the text.

Lastly, we calculate cumulated effects $S_q=\rho'\int_0^\infty e^{-\rho't}\frac{\delta q_t}{q_t}\,dt=$

$$=\rho'(\rho'+g)\int_0^\infty e^{-\rho't}\left(\left(1+\frac{b_k}{\theta-\lambda}\right)T_{rk}\frac{1-e^{-\theta t}}{\theta}-\left(\frac{b_k}{\theta-\lambda}T_{rk}+T_{ra}\right)\frac{1-e^{-\lambda t}}{\lambda}\right)\,dt\frac{\delta A_0}{A_0}=$$

$$=\left(\frac{\rho'+g}{\rho'+\theta}\left(1+\frac{b_k}{\theta-\lambda}\right)T_{rk}-\frac{\rho'+g}{\rho'+\lambda}\left(\frac{b_k}{\theta-\lambda}T_{rk}+T_{ra}\right)\right)\frac{\delta A_0}{A_0}$$

as well as

$$S_w=\rho'\int_0^\infty e^{-\rho't}\frac{\delta w_t}{w_t}\,dt=$$

$$= \rho' \int_0^\infty e^{-\rho' t} \left(- \left(1 + \frac{b_k}{\theta - \lambda} \right) T_{wk} e^{-\theta t} + \left(\frac{b_k}{\theta - \lambda} T_{wk} + T_{wa} \right) e^{-\lambda t} \right) \ dt \delta A_0 / A_0 =$$

$$= \left(- \frac{\rho'}{\rho' + \theta} \left(1 + \frac{b_k}{\theta - \lambda} \right) T_{wk} + \frac{\rho'}{\rho' + \lambda} \left(\frac{b_k}{\theta - \lambda} T_{wk} + T_{wa} \right) \right) \frac{\delta A_0}{A_0}$$

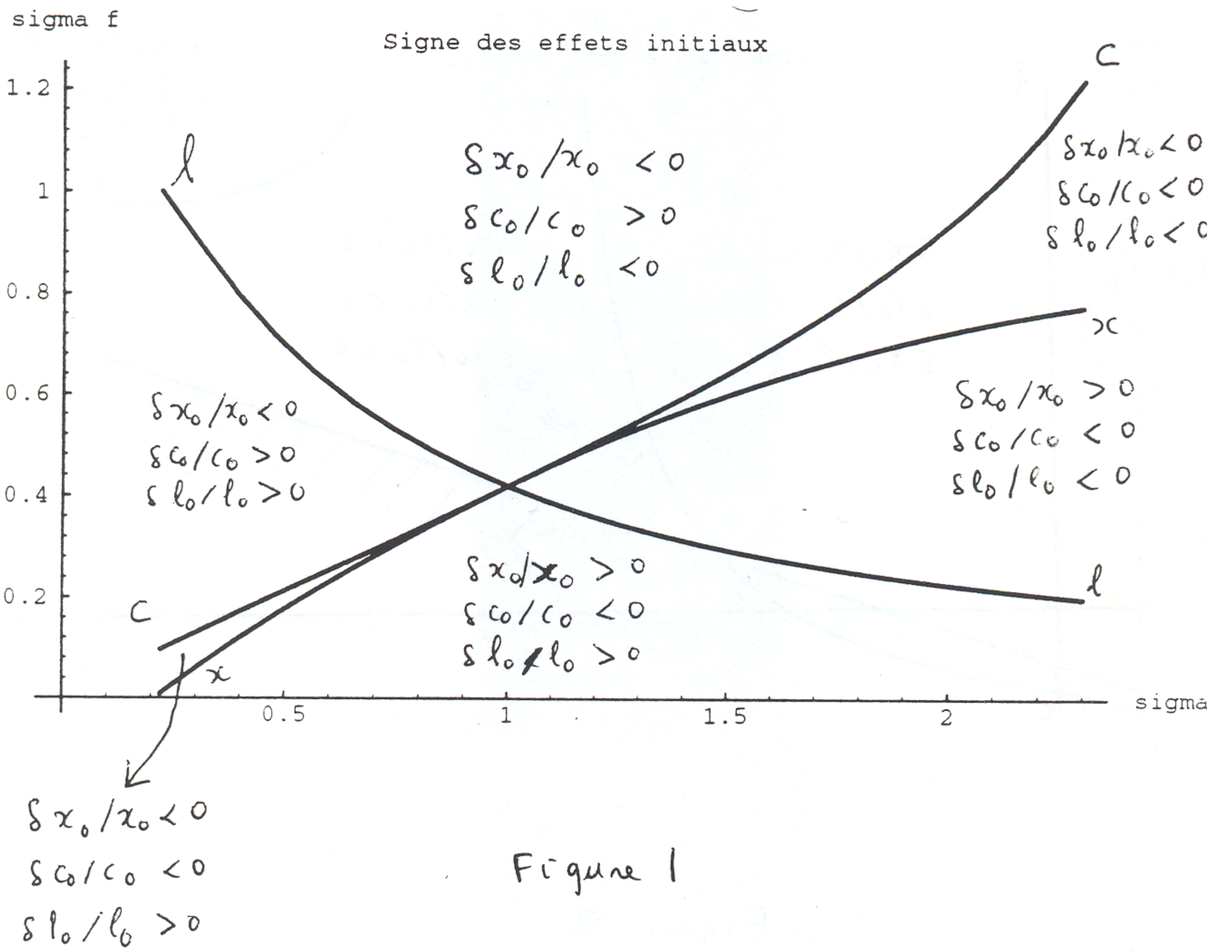
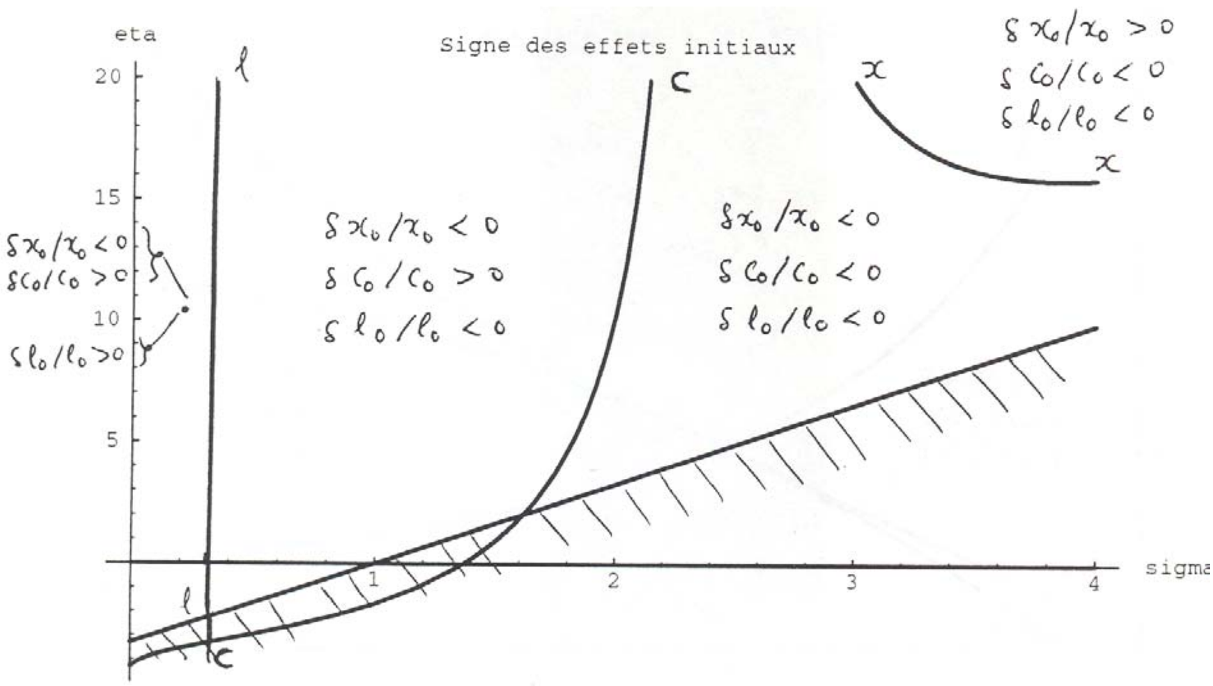
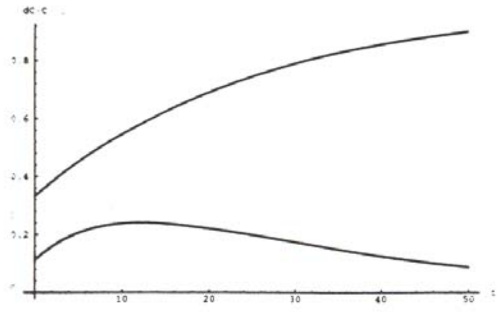

sigma f
Signe des effets initiaux
1.2
1
0.8
0.6
0.4
0.2
0.5
1
1.5
2
sigma
ℓ
c
x
$\delta x_0/x_0 < 0$
$\delta c_0/c_0 > 0$
$\delta \ell_0/\ell_0 < 0$
$\delta x_0/x_0 < 0$
$\delta c_0/c_0 < 0$
$\delta \ell_0/\ell_0 < 0$
$\delta x_0/x_0 < 0$
$\delta c_0/c_0 > 0$
$\delta \ell_0/\ell_0 > 0$
$\delta x_0/x_0 > 0$
$\delta c_0/c_0 < 0$
$\delta \ell_0/\ell_0 < 0$
$\delta x_0/x_0 > 0$
$\delta c_0/c_0 < 0$
$\delta \ell_0/\ell_0 > 0$
$\delta x_0/x_0 < 0$
$\delta c_0/c_0 < 0$
$\delta \ell_0/\ell_0 > 0$

Figure 1

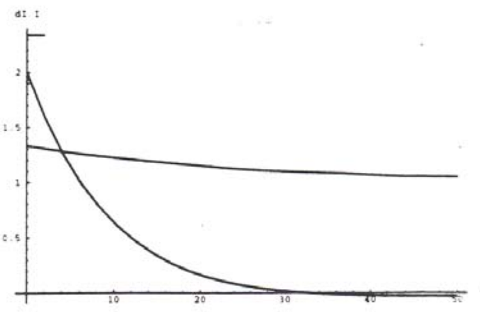


eta
Signe des effets initiaux
20
15
10
5
1
2
3
4
sigma
l
c
x
$\delta x_0/x_0 > 0$
$\delta c_0/c_0 < 0$
$\delta l_0/l_0 < 0$
$\delta x_0/x_0 < 0$
$\delta c_0/c_0 > 0$
$\delta l_0/l_0 < 0$
$\delta x_0/x_0 < 0$
$\delta c_0/c_0 < 0$
$\delta l_0/l_0 < 0$
$\delta x_0/x_0 < 0$
$\delta c_0/c_0 > 0$
$\delta l_0/l_0 > 0$

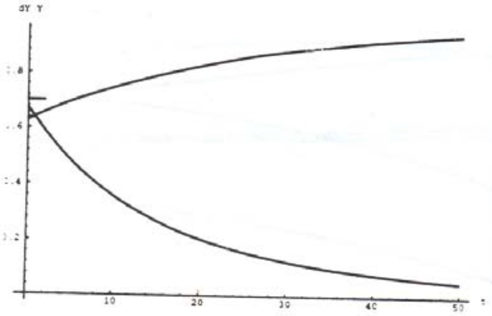
Figure 2



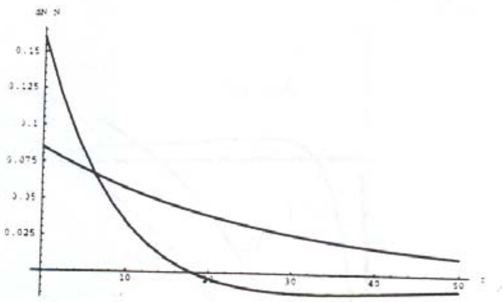
CONSUMPTION



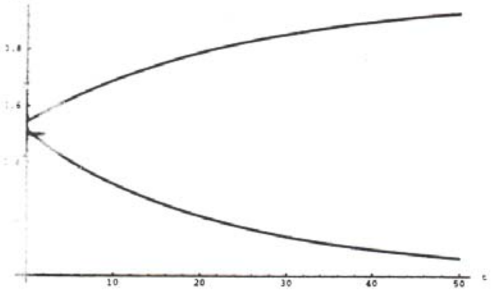
INVESTMENT



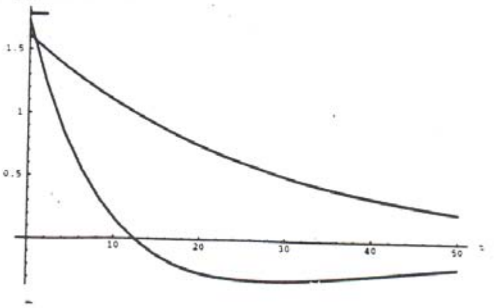
PRODUCTION



LABOR



WAGES



INTEREST RATE

FIGURE 3

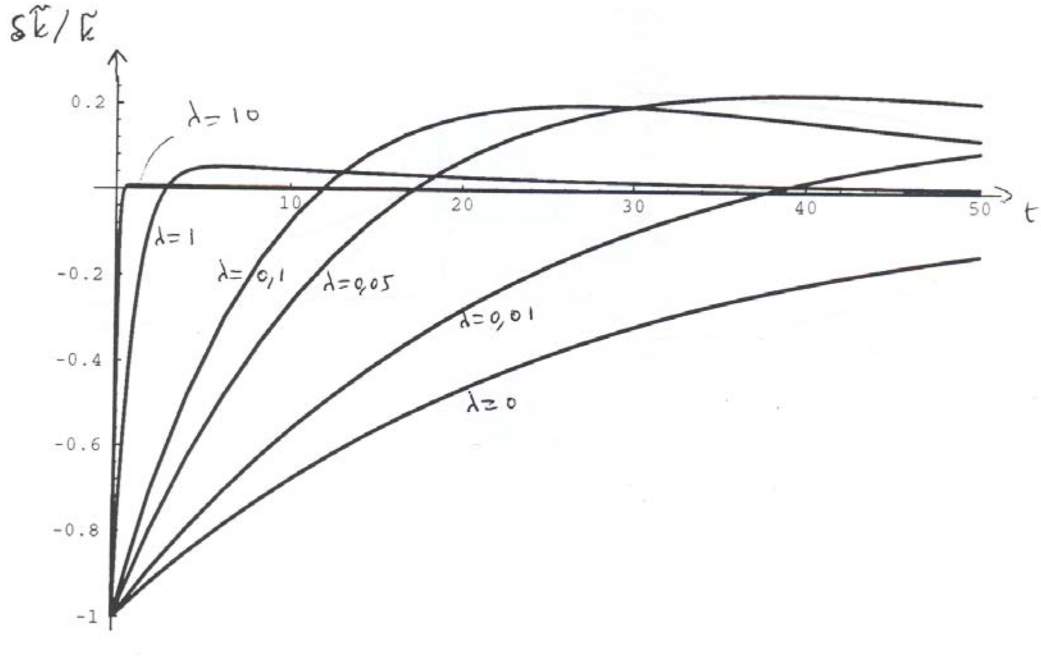
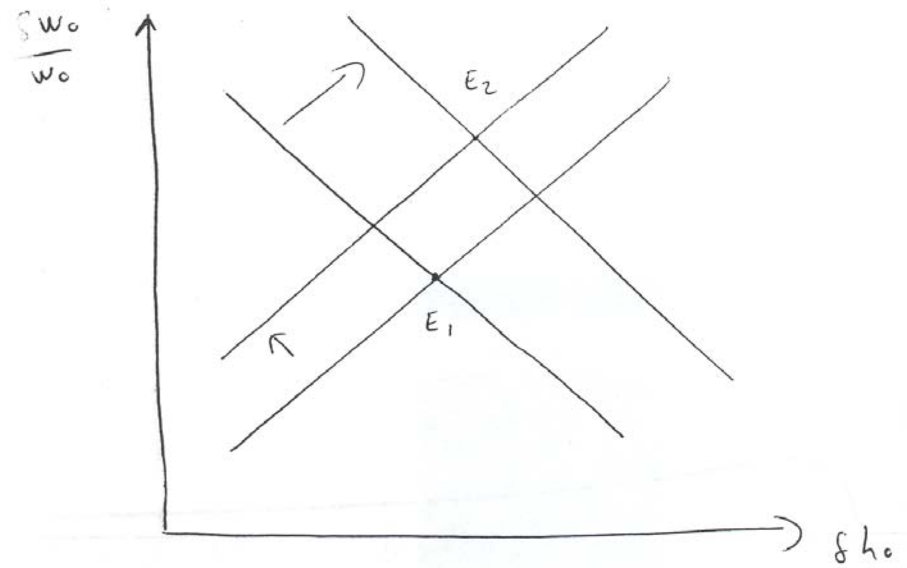
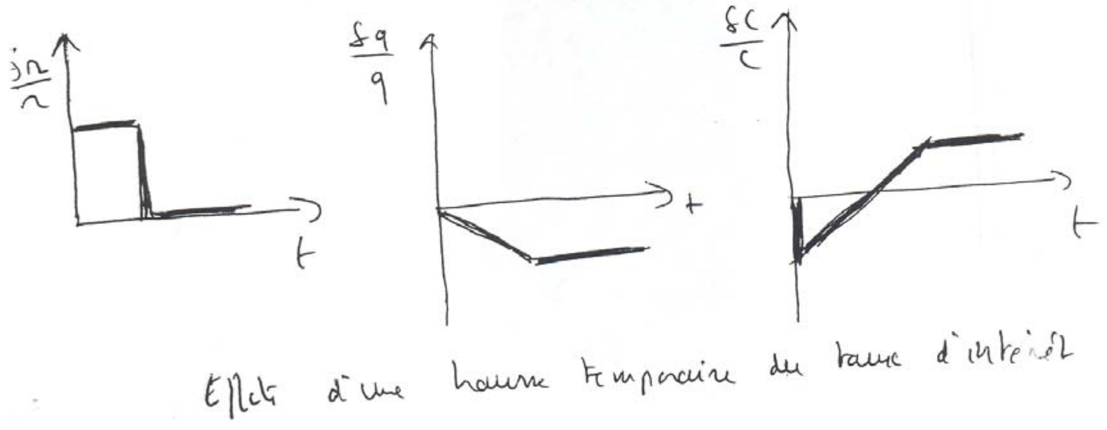


Figure 5



Le marché du travail

Figure 6



Effets d'une hausse temporaire du taux d'intérêt

Figure 7

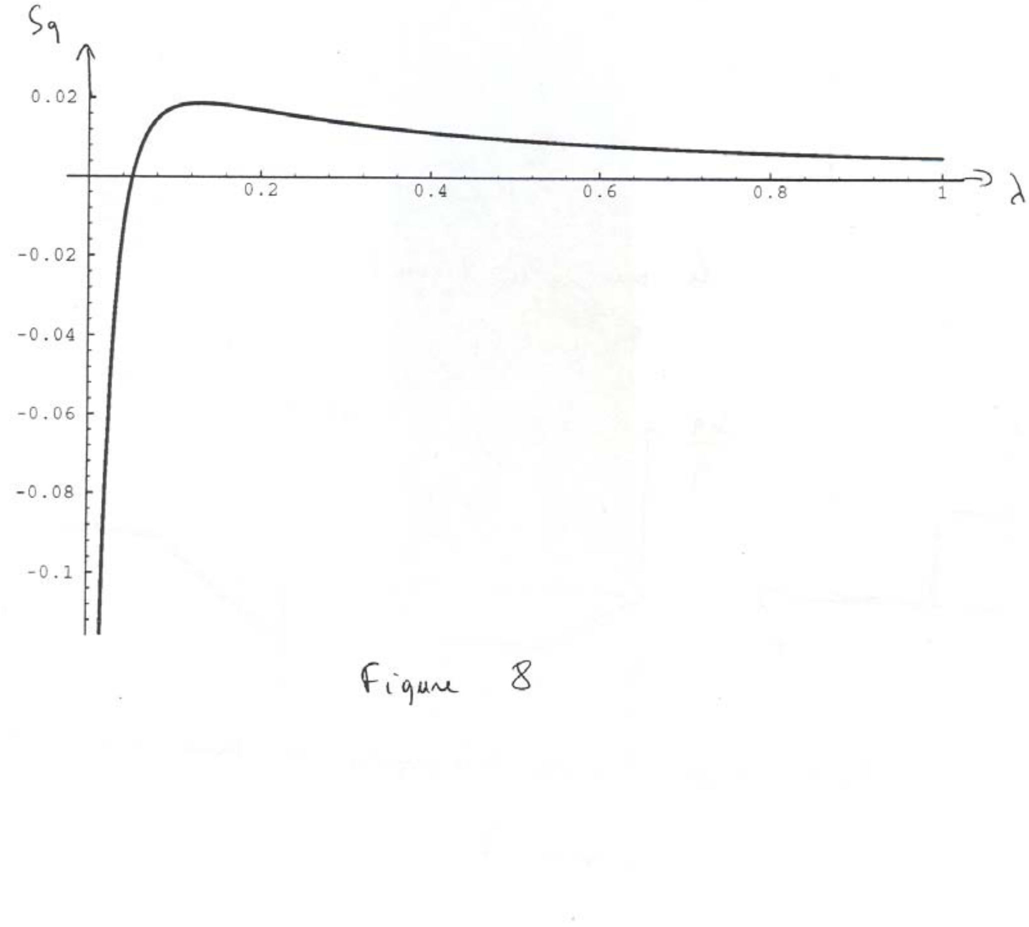


Figure 8